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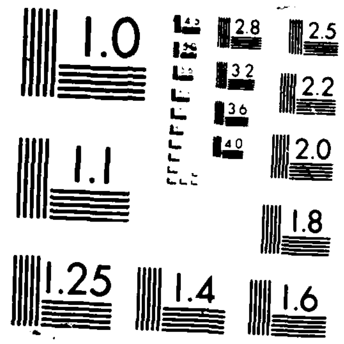

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Spectral Representations of Infinitely Divisible Processes*

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Abstract. The spectral representations for arbitrary discrete parameter infinitely divisible processes as well as for (centered) continuous parameter infinitely divisible processes, which are separable in probability, are obtained. The main tools used for the proofs are (i) a "polar-factorization" of an arbitrary Lévy measure on a separable Hilbert space, and (ii) the Wiener-type stochastic integrals of non-random functions relative to arbitrary "infinitely divisible noise".

0. INTRODUCTION

For the analysis of many statistical and probablistic problems for stationary Gaussian processes, a significant tool is provided by the spectral representations of these processes in terms of the "Gaussian noise". Motivated by these considerations, many authors advocated the need to develop similar spectral representations for symmetric stable processes in terms of the "stable noise" and to apply these to study the analogus problems for these processes; and such representations were in fact developed by several authors (Schilder [**27**], Kuelbs [**13**], Bretagnolle et al [**2**] and Schriber [**28**]). With the same motivation, recently spectral representations of symmetric semistable processes in terms of the "semistable noise" are also obtained (Rajput, Rama-Murthy [**20**]) which are shown to be valid for non-symmetric semistable processes as long as α, the index of the process, is not 1; more recently, a similar result for non-symmetric stable processes with index $\alpha \neq 1$ is also obtained (Hardin [**7**]). Already, the spectral representations of symmetric stable processes have successfully been used to solve the prediction and interpolation problems (e.g. Cambanis, Soltani [**3**], Cambanis, Miamee [**4**], Hosoya [**9**]) and to study the structural and path properties (e.g. Cambanis, Hardin and Weron [**5**], Rootzen [**22**], Rosinski [**25**], and Rosinski and Woyczynski [**26**]) for certain subclasses of these processes.

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In working with Gaussian and symmetric stable processes $X = \{X_t : t\epsilon T\}$ and their spectral representations $\{\int f_t d\Lambda\}$, one discerns two main reasons which make these representations useful in solving various questions about the processes X: (a) Many problems of interest about X can be meaningfully reformulated in terms of the non-random functions f_t and the corresponding "noise" Λ (or sometimes in terms of certain parameters characterizing Λ, e.g. its control measure). (b) These reformulated questions can be effectively solved by making use of the rich structure of the metric linear space of functions generated by $\{f_t\}$ and the fact that Λ enjoys properties very similar to X but, at the same time, admits much simpler probablistic structure. In view of this observation and the remarks made in the previous paragraph, it is thus tempting to suggest that one should develop spectral representations for each subclass of infinitely divisible processes X in terms of the non-random functions f_t belonging to a "nice space" and the "noise" Λ which exhibits properties similar to that of X. But, since different methods of proof may be required to obtain spectral representations for different subclasses of infinitely divisible processes, it may lead to an unending process; and thus a better question would be to ask: Is it possible to develop *one* procedure whereby, for any given infinitely divisible process X, one can choose non-random functions f_t and "an infinitely divisible noise" Λ such that $X \stackrel{d}{=} \{\int f_t d\Lambda\}$ and, additionally, the following criterions are met?

(i) The "noise" Λ retains properties similar to X; for example, if X belongs to a known class such as α-stable or self-decomposable processes, then Λ belongs to the corresponding class of "noises".

(ii) The functions f_t belong to a linear topological space which is "similar" in its structure to that of the linear space of the process X.

The *main theme* of this paper is to provide an "essentially" complete affirmative answer to this question. This is accomplished in two steps: first, we obtain the spectral representations for arbitrary discrete parameter infinitely divisible processes; and then, using this and some limiting arguments, we obtain the representations for continuous parameter infinitely divisible processes which are separable in probability. We reiterate that the representing "noise" Λ and the representing functions f_t chosen for the representations do meet the criterions (i) and (ii), respectively. In fact, as regards to (ii), we show that the space L generated by $\{f_t\}$ is a subspace of a suitable Musielak-Orlicz space, which is continuously (and linearly) embedded in the linear space $L(X)$ of X. Further, if X satisfies some additional conditions (like the ones mentioned above in the continuous parameter case), then we show that L is in fact topologically and linearly isomorphic to $L(X)$. In addition to the above representations which are valid only in law, we also obtain spectral representations which are valid almost surely; this, however, requires that the process be redefined on a slightly larger probability space. Before we end this paragraph we would like to make a few more points: First we note that "integral" representations (in law) of an arbitrary infinitely divisible process in terms of the "Poisson noise" are known (Maruyama [**15**]); but, as neither the noise nor the representing functions necessarily meet the requirements we ask for, these representations do not fall in the category of the spectral representations we are interested in this paper. Second we point out that our spectral representations (in law) of infinitely divisible processes, when specialized to stable and semistable processes, yield, in a unified way, all known spectral representations for these processes mentioned in the first paragraph above. Finally, we mention the papers (Cambanis [**6**], Rajput, Rama-Murthy [**21**] and Hardin [**8**]) which have some relevance to the spectral representations we have discussed above.

Besides the spectral representations noted above, we also present several other results which fall in two broad categories. All of these play a crucial role for our proofs of the spectral representation theorems, but we also feel that these will be of independent interest. In one category of these results, we obtain a "polar factorization" of an arbitrary Lévy measure on ℓ_2 in terms of a finite measure on the boundary of the unit sphere of ℓ_2 and a family of Lévy measures on the real line. This factorization is similar in spirit to the known factorization of a symmetric stable Lévy measure on $\mathbf{R}^n$ (Lévy [**14**]) and on ℓ_2 (Kuelbs [**13**]); and plays an analogus role in the development of the spectral representations here as did the factorization of a symmetric stable Lévy measure for the proofs of the spectral representations of symmetric stable processes in [**2, 13, 27, 28**]. The results in the other category concern with a systematic study of Wiener-type integrals $\int f d\Lambda$ of non-random functions with respect to an arbitrary "infinitely divisible noise" Λ. The main results we present here are: (a) a characterization of Λ-integrable functions in terms of certain parameters of Λ; (b) the identification of the space of Λ-integrable functions as a certain Musielak-Orlicz space; and (c) an isomorphism theorem between this Musielak-Orlicz space and a suitable subspace of L_p-space of random variables. The theory of Wiener-type integrals under various hypotheses on the "noise" Λ has a long history (e.g. Urbanik, Woyczynski [**30**], Urbanik [**29**], Rosinski [**23, 24**], Schilder [**27**] and Rajput and Rama-Murthy [**20**]); the development of these integrals presented here is the most general in the sense that we require minimal hypotheses both on the "noise" Λ and the space on which integrands and Λ are defined.[1]

The organization of the rest of the paper is as follows: Section 1 contains the preliminaries; Section 2 contains the development of stochastic integrals relative to the "infinitely divisible noise" Λ and a characterization of Λ-integrable functions. Section 3 concerns with the identification of the space of Λ-integrable functions as a certain Musielak-Orlicz space and its isomorphism with the subspaces of L_p-space of random variables. Sections 4 and 5 contain, respectively, the spectral representation results (in law) for the discrete and the continuous parameter infinitely divisible processes; Section 4, also contains the "polar factorization" result of Lévy measures on ℓ_2. Section 6 concerns with the spectral representation of infinitely divisible process which hold almost surely. Section 7 constitutes of an appendix and contains a proof of a result which establishes the existence of a measure on the product space given a family of marginal measures satisfying certain hypotheses.

We would take the liberty here to thank Mary Drake for the patience and care she has shown while typing this manuscript.

I. Preliminaries and Some Notations

In this section, we recall some definitions and known facts; also we fix some notations and conventions which we shall use throughout the paper.

Let H be a real (finite or infinite dimensional) separable Hilbert space and let μ be an infinitely divisible (**ID**) prob. measure on H (i.e. μ has a unique n-th root for each $n = 1,2,3\ldots$). As is well known, for every **ID** prob. measure μ, $\{\mu^s : s > o\}$, the set of s-th roots of μ, forms a continuous (in the weak topology) semigroup under convolution, which is also tight on every finite interval of $R^+ = (0,\infty)$. Using this semigroup, we shall now define Gaussian, stable and semistable prob. measures on H. These definitions are non-standard but are equivalent to the traditional definitions which are usually given in terms weak limits of certain normed sums. We adopted this route mainly because we make use of these defining properties of these prob. measures. Before we record these definitions, we introduce a few notations: For a measure ν on H and a nonzero a in R (the reals), we denote by $a\cdot\nu$, the measure defined by $a\cdot\nu(B) = \nu(a^{-1}B)$, for every Borel set B of H;

[1] Recently the authors have received a manuscript by Kwapien and Woyczynski entitled *Semimartingale integrals via decoupling inequalities and tangent processes*. In this paper, they give a characterization of previsible stochastic processes that are integrable relative to semimartingales. As a necessary first step to obtain this result, they also characterize non-random functions that are integrable relative to general "independent increment noise". This later result, obtained independently of ours, has some overlap with our Theorems 3.3 and 3.4 when specialized to $S = [0,\infty)$ and $p = 0$.

further, we shall use the notations $\mathbf{S}(\alpha)$, $\mathbf{S}(r,\alpha)$ and **SD** for the phrases "stable of index α", "semistable of index (r,α)" and "self-decomposable", respectively, where $0 < \alpha < 2$ and $0 < r < 1$. Let now μ be a prob. measure on H, we say μ is a $\mathbf{S}(\alpha)$ (resp. a $\mathbf{S}(r,\alpha)$) *prob. measure* if μ is **ID** and

$$\mu^t = t^{\frac{1}{\alpha}} \cdot \mu * \delta_{x(t)}, \text{ for all } t \in (0,1], \tag{1.1}$$

$$\text{(resp. } \mu^r = r^{\frac{1}{\alpha}} \cdot \mu * \delta_{x(r)}), \tag{1.2}$$

where $\delta_{x(t)}$ and $\delta_{x(r)}$ denote the Dirac measures at the elements $x(t)$ and $x(r)$ of H, respectively, and $*$ denotes the usual convolution operation. If $x(t)$ in (1.1) (resp. in (1.2)) is θ, the zero element of H, and $\alpha \neq 1$, then we say μ is a *centered* $\mathbf{S}(\alpha)$ (resp. a *centered* $\mathbf{S}(r,\alpha)$) *prob. measure.* If $\alpha = 1$, then we say μ is a *centered* $\mathbf{S}(1)$ (resp. a *centered* $\mathbf{S}(r,1)$) *prob. measure* only in the case when μ is a *symmetric* $\mathbf{S}(1)$ (resp. $\mathbf{S}(r,1)$) prob. measure. If μ is **ID** and (1.1) (or equivalently (1.2)) holds with $\alpha = 2$, then we say μ is *Gaussian*, and, if, in addition, $x(t) = \theta$ (or equivalently $x(r) = \theta$), then we say μ is *centered* (or *symmetric*) *Gaussian*. Finally, we say μ is a **SD** *prob. measure*, if

$$\mu = t \cdot \mu * \nu_t, \qquad \text{for all } 0 < t \leq 1, \tag{1.3}$$

where ν_t is a prob. measure on H.

Let now T be an arbitrary index set and $X \equiv \{X_t : t \in T\}$ be a real stochastic process, we say X is an **ID** (resp. a *symmetric* **ID**) *process* if, for every finite set $\{t_1, \cdots, t_n\}$ of T, $\mathcal{L}(X_{t_1}, \cdots, X_{t_n})$, the law of $(X_{t_1}, \cdots, X_{t_n})$, is an **ID** (resp. a symmetric **ID**) prob. measure on $\mathbf{R}^n$, the n-Euclidean space. The definitions of **SD**, $\mathbf{S}(\alpha)$, $\mathbf{S}(r,\alpha)$ and Gaussian processes, of their symmetric counterparts and of centered $\mathbf{S}(\alpha)$ and $\mathbf{S}(r,\alpha)$ processes can be stated in the obvious way.

Now we shall define various **ID** random (r.) measures. Throughout the paper, unless stated otherwise, we denote, by S, an arbitrary non-empty set and, by $\mathcal{S}$, a δ-ring of subsets of S with the property:

(1.4) There exists an increasing sequence $\{S_n\}$ of sets in $\mathcal{S}$ with $\bigcup_n S_n = S$.

Let $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ be a real stochastic process defined on some prob. space $(\Omega, \mathcal{F}, P)$. We call Λ to be an *independently scattered r. measure* (or *r. measure*, for short), if, for every sequence $\{A_n\}$ of disjoint sets in S, the r. variables $\Lambda(A_n)$, $n = 1, 2, \cdots$, are independent, and, if $\bigcup_n A_n$ belong to S, then we also have

$$\Lambda\left(\bigcup_n A_n\right) = \sum_n \Lambda(A_n), \qquad a.s.,$$

where the series is assumed to converge almost surely. In addition, if $\Lambda(A)$ is a symmetric r. variable, for every $A \epsilon S$, then we call Λ a *symmetric r. measure.* We call a r. measure Λ to be an **ID** *r. measure* if $\Lambda(A)$ is **ID**; if, in addition, $\Lambda(A)$ is symmetric, then we call Λ to be a *symmetric* **ID** *r. measure.* The definitions of $\mathbf{S}(\alpha)$, $\mathbf{S}(r, \alpha)$, **SD** and Gaussian r. measures, of their symmetric counterparts and of centered $\mathbf{S}(\alpha)$ and $\mathbf{S}(r, \alpha)$ r. measures can be stated analogously.

Before we end this section, we would like to mention a few more conventions and notations: While writing the Lévy representation of the characteristic (ch.) function $\hat{\mu}$ of an **ID** prob. measure μ on H one can use many different centering functions, we found the centering function

$$\tau(z) = \left\{ \begin{array}{ll} ||z|| & \text{if} \quad ||z|| \leq 1 \\ \dfrac{z}{||z||} & \text{if} \quad ||z|| > 1 \end{array} \right\}$$

easier to work with in our calculations. We shall, therefore, use this centering function throughout. By a Lévy measure defined on a Borel subset B of H, we shall always mean any measure M on B satisfying $\int_B \min(1, ||z||^2)\, dM < \infty$, with $M(\{\theta\}) = 0$, if $\theta \epsilon B$. Whenever it is important that M be defined on the whole of H, we will do so by assigning $M(B^c) = 0$; but will use the same notation for the extended measure.

By the statement "M is a **SD** Lévy measure on B" we would mean that M is a Lévy measure of a **SD** prob. measure on H; we shall adopt a similar convention relative to the Levy measures of other classes of **ID** prob. measures on H. Finally, for a given topological space X, $\mathcal{B}(X)$ will always denote its Borel σ-algebra.

II. Infinitely Divisible Random Measures and Stochastic Integrals

Throughout this paper $\Lambda = \{\Lambda(A) : A \epsilon S\}$ will denote an **ID** r. measure defined on some prob. space $(\Omega, \mathcal{F}, P)$ (recall that S stands for a δ-ring of subsets of an arbitrary non-empty set S satisfying (1.4)). Since, for every $A \epsilon S$, $\Lambda(A)$ is an **ID** r. variable, its ch. function can be written in the Lévy's form:

$$(2.1) \qquad \hat{\mathcal{L}}(\Lambda(A))(t) = \exp\left\{ it\nu_0(A) - \frac{1}{2}t^2\nu_1(A) + \int_{\mathbf{R}} (e^{itx} - 1 - it\tau(x)) F_A(dx) \right\},$$

where $-\infty < \nu_0(A) < \infty$, $0 \leq \nu_1(A) < \infty$ and F_A is a Lévy measure on $\mathbf{R}$. In this section, we first show (Proposition 2.1) that there is a one to one correspondence between the class of **ID** r. measures on one hand and the class of parameters ν_0, ν_1 and $F_.$ on the other. This fact, under various additional assumptions, was "essentially" proved in Prékopa [**18, 19**] and Urbanik and Woyczynski [**30**]. We include a proof of this fact here, since this proposition is quite important to us and since our proof is very simple and uses only standard arguments of the classical probability theory. Through this result we also define λ, the control measure of Λ. Next we show (Lemma 2.3) that $F_.(\cdot)$ determines a unique measure on $\sigma(\mathbf{S}) \times \mathcal{B}(\mathbf{R})$ which admits a factorization in terms of a family of Lévy measures $\rho(s,\cdot)$, $s \epsilon S$ on $\mathbf{R}$ and the measure λ. This fact plays an important role throughout the paper; in particular, this helps us derive another form of the ch. function of $\mathcal{L}(\Lambda(A))$ in terms of the measures $\rho(s,\cdot)$ and λ (Proposition 2.5). This form of the ch. function plays a crucial role in obtaining the ch. function of the stochastic integral $\int_S f d\Lambda$ (which we also define) (Proposition 2.6) and in the proof of the *main result* of this section (Theorem 2.7) which provides an important characterization of Λ-integrable functions.

PROPOSITION 2.1. *(a) Let Λ be an* **ID** *r. measure with the ch. function given by (2.1). Then $\nu_0 : S \longmapsto \mathbf{R}$ is a signed-measure, $\nu_1 : S \longmapsto [0,\infty)$ is a measure, F_A is a Lévy measure on $\mathbf{R}$, for every $A \epsilon S$, and $S \ni A \longmapsto F_A(B) \epsilon [0,\infty)$ is a measure, for every $B \epsilon \mathcal{B}(\mathbf{R})$, whenever $0 \notin \overline{B}$.*

(b) Let ν_0, ν_1 and $F_.$ satisfy the conditions given in (a). Then there exists a unique (in the sense of finite-dimensional distributions) **ID** *r. measure Λ such that (2.1) holds.*

(c) Let ν_0, ν_1 and $F_.$ be as in (a) and define

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_{\mathbf{R}} \min\{1, x^2\} F_A(dx), \qquad A \epsilon S.$$

Then $\lambda : S \longmapsto [0,\infty)$ is a measure such that $\lambda(A_n) \longmapsto 0$ implies $\Lambda(A_n) \longrightarrow 0$ in prob. for every $\{A_n\} \subset S$; further, if $\Lambda(A'_n) \longrightarrow 0$ in prob. for every sequence $\{A'_n\} \subset S$ such that $A'_n \subset A_n \epsilon S$, then $\lambda(A_n) \longrightarrow 0$.

PROOF: (a) Let $\{A_k\}_{k=1}^n$ be pairwise disjoint sets in S. By the uniqueness of Lévy's representation of the ch. function of an **ID** distribution, it follows, using $\hat{\mathcal{L}}\left(\Lambda\left(\bigcup_{k=1}^n A_k\right)\right) = \prod_{k=1}^n \hat{\mathcal{L}}\left(\Lambda\left(A_k\right)\right)$, that all three set functions ν_0, ν_1 and $F_.(B)$ are finitely additive. Let now $A_n \epsilon S$, $A_n \searrow \emptyset$. Since $\Lambda(A_n) \longrightarrow 0$ in prob., we have that $\nu_0(A_n) \to 0$, $\nu_1(A_n) \to 0$ and $\int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) \to 0$. By Chebychev's inequality, we get

$$F_{A_n}(\{|x| \geq \epsilon\}) \leq \epsilon^{-2} \int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) \to 0,$$

for every $\varepsilon \epsilon (0,1)$, which completes the proof of (a).

(b) The existence of a finitely additive independently scattered r. measure $\Lambda = \{\Lambda(A) : A \epsilon \mathcal{S}\}$ follows by a standard application of the Kolmogorov Extension Theorem (see e.g. [**11**]). To prove that Λ is countably additive, let $A_n \epsilon \mathcal{S}$, $A_n \searrow \emptyset$. Since $F_{A_1} \geq F_{A_2} \geq \cdots$, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) &\leq \overline{\lim_{n\to\infty}} \int_{\{|x|<\varepsilon\}} \min\{1, x^2\} F_{A_n}(dx) \\ &\quad + \overline{\lim_{n\to\infty}} F_{A_n}(\{|x| \geq \varepsilon\}) \\ &\leq \int_{\{|x|<\varepsilon\}} \min\{1, x^2\} F_{A_1}(dx) \ ,\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Letting $\varepsilon \to 0$ we obtain that $\int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) \to 0$. Since also $\nu_0(A_n) \to 0$ and $\nu_1(A_n) \to 0$, we get $\Lambda(A_n) \to 0$ in prob., proving that Λ is countably additive.

(c). It follows that λ is countably additive by a similar argument as we used for proving the countable additivity of Λ above. For the last part, decompose $A_n = A_n^{(1)} \cup A_n^{(2)}$ such that $\nu_0\left(A_n^{(1)}\right) = \nu_0^+(A_n)$ and $\nu_0(A_n^{(2)}) = -\nu_0^-(A_n)$. Since $\Lambda\left(A_n^{(i)}\right) \to 0$ in prob. as $n \to \infty$, $i = 1, 2$, we get that $\nu_0\left(A_n^{(i)}\right) \to 0$, $\nu_1\left(A_n^{(i)}\right) \to 0$ and $\int_{\mathbf{R}} \min\{1, x^2\} F_{A_n^{(i)}}(dx) \to 0$ as $n \to \infty$, $i = 1, 2$. This implies that $\lambda(A_n) \to 0$. ∎

Definition 2.2. Since $\lambda(S_n) < \infty$, $n = 1, 2, \ldots$ we may (and do) extend λ to a σ-finite measure on $(S, \sigma(\mathcal{S}))$; we call λ, the *control measure* of Λ.

LEMMA 2.3. *Let F. be as in Proposition 2.1(a). Then there exists a unique σ-finite measure F on $\sigma(\mathcal{S}) \times \mathcal{B}(\mathbf{R})$ such that*

$$F(A \times B) = F_A(B), \text{ for all } A \epsilon \mathcal{S},\ B \epsilon \mathcal{B}(\mathbf{R})\,.$$

Moreover, there exists a function $\rho : S \times \mathcal{B}(\mathbf{R}) \longmapsto [0, \infty]$ such that

(i) *$\rho(s, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbf{R})$, for every $s \epsilon S$,*

(ii) *$\rho(\cdot, B)$ is a Borel measurable function, for every $B \epsilon \mathcal{B}(\mathbf{R})$,*

(iii) *$\int_{S\times\mathbf{R}} h(s, x) F(ds,\, dx) = \int_S \left[\int_{\mathbf{R}} h(s, x) \rho(s, dx)\right] \lambda(ds)$, for every $\sigma(\mathcal{S}) \times \mathcal{B}(\mathbf{R})$-measurable function $h : S \times \mathbf{R} \to [0, \infty]$. This equality can be extended (with obvious restrictions regarding the arithmetic of $\pm\infty$) to real and complex-valued functions h.*

The proof of Lemma 2.3 relies on some measure-theoretic facts stated in Proposition 2.4. This proposition, under additional assumptions on the space T and the function Q_0 is essentially known (one proves the existence of Q by using standard compactness argument for the extension of measures; and, then one constructs $q(\cdot,\cdot)$, using the existence of the regular conditional prob. on $T \times \mathbf{R}$). Since in our paper $(T, \mathcal{A})$ is an arbitrary measurable space, the compactness argument cannot be used; and a ***basically*** different proof seems to be required to construct the measure Q. In fact, we first construct the function $q(\cdot,\cdot)$ and then use Tulcea's Theorem ([**1**] p. 209) to construct Q. We are unable to find a reference for a proof of this result in the present general form, for this reason and for reasons of completeness, we include a proof in Appendix (Section 7). We separated this proof from the main body of the paper, since it does not provide any insight to the main ideas of this paper.

PROPOSITION 2.4. *Let $(X, \mathcal{B})$ be a standard Borel space (i.e. a measurable space such that $\mathcal{B}$ is σ-isomorphic to the Borel σ-algebra of some complete separable metric space), and let $(T, \mathcal{A})$ be an arbitrary measurable space. Let $Q_0(A, B)$ be a non-negative function of $A \epsilon \mathcal{A}$, $B \epsilon \mathcal{B}$, satisfying:*

(a) *for every $A \epsilon \mathcal{A}$, $Q_0(A, \cdot)$ is a measure on $(X, \mathcal{B})$,*
(b) *for every $B \epsilon \mathcal{B}$, $Q_0(\cdot, B)$ is a measure on $(T, \mathcal{A})$,*
(c) *the measure λ_0 defined by $\lambda_0(A) = Q_0(A, X)$ is σ-finite on $(T, \mathcal{A})$.*

Then there exists a unique measure Q on the product σ-algebra $\mathcal{A} \times \mathcal{B}$ such that

$$Q(A \times B) = Q_0(A, B) = \int_A q(t, B)\lambda_0(dt)\,, \tag{2.2}$$

for every $A \epsilon \mathcal{A}$, $B \epsilon \mathcal{B}$, where $q : T \times \mathcal{B} \to [0, 1]$ fulfills the following conditions:

(d) *for every t, $q(t, \cdot)$ is a probability measure on $\mathcal{B}$,*
(e) *for every B, $q(\cdot, B)$ is $\mathcal{A}$-measurable.*

Further, if $q_1(\cdot,\cdot)$ is some other function satisfying (2.2), (d) and (e), then off a set of λ_0-measure zero, $q_1(t, \cdot) = q(t, \cdot)$.

PROOF OF LEMMA 2.3: Put

$$G_0(A, B) = \int_B \min\{1, x^2\} F_A(dx),\ \ A \epsilon S,\ \ B \epsilon \mathcal{B}(\mathbf{R})\,.$$

Since for every $B \epsilon \mathcal{B}(\mathbf{R})$, $G_0(\cdot, B)$ is a finite measure on $(S_n, S \cap S_n)$, $n \geq 1$, $G_0(\cdot, B)$ has a unique extension to a σ-finite measure on $(S, \sigma(S))$. Denoting this extension by $Q_0(A, B)$, we see that the assumptions of Proposition 2.4 are satisfied with $(T, \mathcal{A}) = (S, \sigma(S))$ and

$(X, \mathcal{B}) = (\mathbf{R}, \mathcal{B}(\mathbf{R}))$. Thus there exists a measure Q on the product σ-algebra $\sigma(S) \times \mathcal{B}(\mathbf{R})$ such that

$$Q(A \times B) = G_0(A, B) = \int_A q(s, B)\lambda_0(ds) ,$$

where $\lambda_0(A) = G_0(A, \mathbf{R})$ and q satisfies (d) and (e) of Proposition 2.4. Note that $\lambda_0(A) \leq \lambda(A)$, for every $A \epsilon \sigma(S)$, which implies that $\lambda_0 \ll \lambda$; now define

$$\rho(s, dx) = \frac{d\lambda_0}{d\lambda}(s) \left(\min\{1, x^2\}\right)^{-1} q(s, dx) .$$

Then (ii) is satisfied and

$$\int_{\mathbf{R}} \min\{1, x^2\}\rho(s, dx) = \frac{d\lambda_0}{d\lambda}(s) \int_{\mathbf{R}} q(s, dx) = \frac{d\lambda_0}{d\lambda}(s) \leq 1 ,$$

which proves (i) (we may always assume that $\frac{d\lambda_0}{d\lambda}(s) \leq 1$ for all s). Define

$$F(C) = \int_S \left[\int_{\mathbf{R}} I_C\left((s, x)\right) \rho(s, dx)\right] \lambda(ds) , \tag{2.3}$$

$C \epsilon \sigma(S) \times \mathcal{B}(\mathbf{R})$; then F is a well-defined measure that satisfies, for every $A \epsilon S$ and $B \epsilon \mathcal{B}(\mathbf{R})$,

$$\begin{aligned} F(A \times B) &= \int_A \left[\int_B \rho(s, dx)\right] \lambda(ds) \\ &= \int_A \left[\int_B \left(\min\{1, x^2\}\right)^{-1} q(s, dx)\right] \lambda_0(ds) \\ &= \int_{A \times B} \left(\min\{1, x^2\}\right)^{-1} Q(ds, dx) \\ &= \int_B \left(\min\{1, x^2\}\right)^{-1} G_0(A, dx) = F_A(B); \end{aligned}$$

(iii) now follows from (2.3) by a standard argument. This completes the proof of Lemma 2.3. ▮

Using Lemmas 2.1 and 2.3 we obtain a very useful form of the ch. function of $\Lambda(A)$:

PROPOSITION 2.5. *The ch. function (2.1) of $\Lambda(A)$ can be rewritten in the form:*

$$\hat{\mathcal{L}}\left(\Lambda(A)\right)(t) = \exp\{\int_A K(t, s)\lambda(ds)\}, \ t \epsilon \mathbf{R}, \ A \epsilon S ,$$

where

$$K(t, s) = ita(s) - \frac{1}{2}t^2\sigma^2(s) + \int_{\mathbf{R}} \left(e^{itx} - 1 - it\tau(x)\right) \rho(s, dx),$$

$a(s) = \frac{d\nu_0}{d\lambda}(s)$, $\sigma^2(s) = \frac{d\nu_1}{d\lambda}(s)$ and ρ is given by Lemma 2.3. Moreover, we have

$$(2.4) \qquad |a(s)| + \sigma^2(s) + \int_{\mathbf{R}} \min\{1, x^2\}\rho(s, dx) = 1 \qquad a.e.[\lambda].$$

PROOF: First part immediately follows from (2.1) and Lemma 2.3. Since, for every $A \epsilon S$, we have

$$\int_A \left[|a(s)| + \sigma^2(s) + \int_{\mathbf{R}} \min\{1, x^2\}\rho(s, dx)\right] \lambda(ds) =$$
$$|\nu_0|(A) + \nu_1(A) + \int_{S \times \mathbf{R}} \min\{1, x^2\} F(ds, dx) = \lambda(A) = \int_A d\lambda\,,$$

(2.4) follows; which completes the proof. ∎

The following definition of the stochastic integral, proposed first by Urbanik and Woyczynski [**30**] is the usual definition of the integrals with respect to a vector measure taking values in the $L_0(\Omega, \mathcal{F}, P)$-space (see also [**23**]).

Definition. (a) Let $f = \sum_{j=1}^n x_j I_{A_j}$ be a real simple function on S, where $A_j \epsilon S$ are disjoint. Then, for every $A \epsilon \sigma(S)$, we define

$$\int_A f d\Lambda = \sum_{j=1}^n x_j \Lambda(A \cap A_j)\,.$$

(b) A measurable function $f : (S, \sigma(S)) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is said to be Λ-*integrable* if there exists a sequence $\{f_n\}$ of simple functions as in (a) such that

(i) $f_n \to f$ a.e. $[\lambda]$,

(ii) for every $A \epsilon \sigma(S)$, the sequence $\{\int_A f_n d\Lambda\}$ converges in prob., as $n \to \infty$.

If f is Λ-integrable, then we put

$$\int_A f d\Lambda = P - \lim_{n \to \infty} \int_A f_n d\Lambda\,,$$

where $\{f_n\}$ satisfies (i) and (ii).

We note that $\int_A f d\Lambda$ is well defined (i.e. it does not depend on the approximating sequence $\{f_n\}$, Urbanik and Woyczynski [**30**]). Now we proceed to find an expression of the ch. function of $\int_S f d\Lambda$:

PROPOSITION 2.6. *If f is Λ-integrable, then $\int_S |K(tf(s),s)|\lambda(ds) < \infty$, where K is given in Proposition 2.5, and*

$$\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(t) = \exp\left\{\int_S K(tf(s),s)\lambda(ds)\right\}, \qquad t\epsilon\mathbf{R}. \tag{2.5}$$

PROOF: Note first that (2.5) holds for simple functions. Let $\{f_n\}$ be a sequence of simple functions in the definition of Λ-integral. Define complex measures $\mu_{t,n}$, $t\epsilon\mathbf{R}$, $n \geq 1$, by

$$\mu_{t,n}(A) = \int_A K(tf_n(s),s)\lambda(ds), \qquad A\epsilon\sigma(S).$$

Since, for every $t\epsilon R$ and $A\epsilon\sigma(S)$,

$$\begin{aligned}\lim_{n\to\infty}\mu_{t,n}(A) &= \lim_{n\to\infty}\log\hat{\mathcal{L}}\left(\int_A f_n d\Lambda\right)(t)\\ &= \log\hat{\mathcal{L}}\left(\int_A f d\Lambda\right)(t)\\ &= \mu_t(A),\end{aligned}$$

it follows, by the Hahn-Saks-Vitali Theorem, that μ_t is a countably additive complex measure. Clearly μ_t is absolutely continuous with respect to λ. Therefore, for every $t\epsilon R$, there exists an $h_t\epsilon L_1(S,\sigma(S),\lambda;\mathbf{C})$ such that

$$\log\hat{\mathcal{L}}\left(\int_A f d\Lambda\right)(t) = \mu_t(A) = \int_A h_t(s)\lambda(ds),$$

for every $A\epsilon\sigma(S)$. To end the proof it suffices to show that $h_t(s) = K(tf(s),s)$ a.e. $[\lambda]$, for each $t\epsilon\mathbf{R}$. Let $t\epsilon\mathbf{R}$ be fixed. By the continuity of $K(\cdot,s)$, for each $s\epsilon S$, we obtain

$$K(tf_n(s),s) \to K(tf(s),s) \text{ a.e. } [\lambda], \tag{2.6}$$

as $n\to\infty$. Using Egorov's Theorem, we may decompose S as follows: $S = \bigcup_{j=0}^{\infty} A_j$, where $\lambda(A_0) = 0$, $\lambda(A_j) < \infty$, if $j \geq 1$, and such that (2.6) holds uniformly in $s\epsilon A_j$, $j = 1,2,\ldots$. Hence, for every $j \geq 1$ and $A\epsilon\sigma(S)$,

$$\begin{aligned}\int_{A\cap A_j} h_t(s)\lambda(ds) = \mu_t(A\cap A_j) &= \lim_{n\to\infty}\int_{A\cap A_j} K(tf_n(s),s)\lambda(ds)\\ &= \int_{A\cap A_j} K(tf(s),s)\lambda(ds).\end{aligned}$$

It follows that $h_t(s) = K(tf(s),s)$ *a.e.*$[\lambda]$ on A_j, $j \geq 1$. Since A_0 is a λ-null set, the last equality holds *a.e.*$[\lambda]$ on S. ∎

As we noted in the beginning of this section, the following is the main result of this section. It provides a necessary and sufficient condition for the existence of $\int_S f d\Lambda$ in terms of the deterministic characteristics of Λ.

THEOREM 2.7. *Let $f : S \to \mathbf{R}$ be a $\sigma(S)$-measurable function. Then f is Λ-integrable if and only if the following three conditions hold:*

(i) $\int_S |U(f(s), s)|\lambda(ds) < \infty$,

(ii) $\int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty$,

and

(iii) $\int_S V_0(f(s), s)\lambda(ds) < \infty$,

where

$$U(u, s) = ua(s) + \int_R (\tau(xu) - u\tau(x))\,\rho(s, dx),$$

$$V_0(u, s) = \int_R \min\{1, |xu|^2\}\,\rho(s, dx).$$

Further, if f is Λ-integrable, then the ch. function of $\int_S f d\Lambda$ can be written as

(iv) $\hat{\mathcal{L}}(\int_S f d\Lambda)(t) = \exp\left\{ita_f - \frac{1}{2}t^2\sigma_f^2 + \int_R(e^{itx} - 1 - it\tau(x))F_f(dx)\right\}$,

where

$$a_f = \int_S U(f(s), s)\lambda(ds),$$

$$\sigma_f^2 = \int_S |f(s)|^2\sigma^2(s)\lambda(ds),$$

and

$$F_f(B) = F(\{(s, x)\epsilon S \times \mathbf{R} : f(s)x\epsilon B \setminus \{0\}\}), \quad B\epsilon\mathcal{B}(\mathbf{R}).$$

PROOF: Assume that f is Λ-integrable. By Proposition 2.6, we have that

$$\begin{aligned}\left|\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(t)\right|^2 &= \exp\left\{2\int_S ReK(tf(s), s)\lambda(ds)\right\}\\ &= \exp\left\{2\int_S\left[-\frac{1}{2}t^2f^2(s)\sigma^2(s) + \int_R(\cos(tf(s)x) - 1)\,\rho(s, dx)\right]\lambda(ds)\right\}\\ &= \exp\left\{-t^2\sigma_f^2 + 2\int_R(\cos tx - 1)F_f(dx)\right\}\end{aligned}$$

is the ch. function of an **ID** distribution. Hence $\sigma_f^2 < \infty$ and $\int_R \min\{1, x^2\}F_f(dx) < \infty$. This proves (ii) and (iii). Now, since $|\tau(x) - \sin x| \le 2\min\{1, x^2\}$, we get

$$\begin{aligned}|U(u, s)| &\le \left|ua(s) + \int_R[\sin xu - u\tau(x)]\,\rho(s, dx)\right|\\ &\quad + \left|\int_R[\tau(xu) - \sin xu]\,\rho(s, dx)\right|\\ &\le |ImK(u, s)| + 2V_0(u, s).\end{aligned}$$

Thus (i) follows by Proposition 2.6 and already proven (iii). In view of (i), (ii) and (iii), it is easy to derive (iv) from (2.5).

Conversely, assume that (i), (ii) and (iii) hold. Let $A_n = \{s : |f(s)| \le n\} \cap S_n$. We have that $A_n \epsilon \mathcal{S}$ and $A_n \nearrow S$. Choose f_n's, simple $\mathcal{S}$-measurable functions, such that $f_n(s) = 0$, if $s \notin A_n$, $|f_n(s) - f(s)| \le \frac{1}{n}$, if $s \epsilon A_n$, and $|f_n(s)| \le |f(s)|$, for all $s \epsilon S$. Clearly $f_n \to f$ everywhere on S, as $n \to \infty$. Since, for every $A \epsilon \sigma(\mathcal{S})$ and $n,\ m \ge 1$,

$$|[f_n(s) - f_m(s)]\, 1_A(s)| \le 2|f(s)|,$$

by Lemma 2.8, which follows this proof, we get

$$|U([f_n(s) - f_m(s)]\, 1_A(s), s)| \le 2\,|U(f(s), s)| + 27V_0\,(f(s), s)\ .$$

Therefore, by the Dominated Convergence Theorem, we obtain that, for every $A \epsilon \sigma(\mathcal{S})$,

$$\lim_{n,m\to\infty} \int_S U([f_n(s) - f_m(s)]\, 1_A(s), s)\, \lambda(ds) = 0\,,$$
$$\lim_{n,m\to\infty} \int_S [f_n(s) - f_m(s)]^2\, 1_A(s)\sigma^2(s)\lambda(ds) = 0\,,$$

and

$$\lim_{n,m\to\infty} \int_S V_0([f_n(s) - f_m(s)]\, 1_A(s), s)\, \lambda(ds) = 0\,.$$

In view of (iv), $\lim_{n,m\to\infty} \hat{\mathcal{L}}\left(\int_S [f_n - f_m]\, 1_A d\Lambda\right)(t) \to 1$, for every $t \epsilon \mathbf{R}$ and $A \epsilon \sigma(\mathcal{S})$. Hence the sequence $\left\{\int_A f_n d\Lambda\right\}_{n=1}^{\infty}$ converges in prob., for every $A \epsilon \sigma(\mathcal{S})$; i.e. f is Λ-integrable. ∎

LEMMA 2.8. *For every $u \epsilon \mathbf{R}$, $s \epsilon S$ and $d > 0$,*

$$\sup\{|U(cu, s)| : |c| \le d\} \le d\,|U(u, s)| + (1 + d)^3 V_0(u, s).$$

PROOF: Let $|c| \le d$. We have

$$\begin{aligned} U(cu, s) &= cua(s) + \int_{\mathbf{R}} [\tau(cux) - cu\tau(x)]\, \rho(s, dx) \\ &= cua(s) + c\int_{\mathbf{R}} [\tau(ux) - u\tau(x)]\, \rho(s, dx) \\ &\qquad + \int_{\mathbf{R}} [\tau(cux) - c\tau(ux)]\, \rho(s, dx) \\ &= cU(u, s) + R(c, u, s)\,, \end{aligned}$$

where $R(c,u,s)$ denotes the last integral. Since $\tau(cux) - c\tau(ux) = 0$ if $|ux| \leq \min\{1, |c|^{-1}\}$ and $|\tau(cux) - c\tau(ux)| \leq 1 + d$ otherwise, we get

$$\begin{aligned}
|R(c,u,s)| &\leq (1+d) \int_{\{|ux|>\min\{1,|c|^{-1}\}\}} \rho(s,dx) \\
&\leq (1+d)\rho\left(s, \{x : \min\{1, |ux|\} \geq \min\{1, |c|^{-1}\}\}\right) \\
&\leq \frac{1+d}{\min\{1, |c|^{-2}\}} \int_{\mathbf{R}} \min\{1, |ux|^2\}\rho(s,dx),
\end{aligned}$$

by Chebyshev's inequality. Since the last quantity is bounded by $(1+d)^3 V_o(u,s)$, the proof is complete. ∎

Usually it is easier to verify conditions for the existence of $\int f d\Lambda$ when Λ is symmetric. The next proposition shows how to characterize the Λ-integrable functions f, using $\overline{\Lambda}$-integrability of f, where $\overline{\Lambda}$ is the symmetrization of Λ.

PROPOSITION 2.9. *Let Λ' be an independent copy of Λ and put $\overline{\Lambda}(A) = \Lambda(A) - \Lambda'(A)$, $A \epsilon S$. Then for an arbitrary function $f : S \longmapsto \mathbf{R}$, f is Λ-integrable if an only if f is $\overline{\Lambda}$-integrable and the condition (i) of Theorem 2.7 is fulfilled.*

PROOF: The Lemma follows immediately from Theorem 2.7 because

$$\hat{\mathcal{L}}\left(\overline{\Lambda}(A)\right)(t) = \exp\left\{\int_A \left[-t^2\sigma^2(s) + 2\int_{\mathbf{R}} (\cos tx - 1)\overline{\rho}(s,dx)\right] \lambda(ds)\right\},$$

where $\overline{\rho}(s,B) = \rho(s,B) + \rho(s,-B)$, $B \epsilon \mathcal{B}(\mathbf{R})$. ∎

III. Continuity of the Stochastic Integral Mapping and Identification of Λ-integrable Functions

In this section we shall identify the set of Λ-integrable functions as a certain Musielak-Orlicz modular space, and shall prove the continuity of the mapping $f \to \int_S f d\Lambda$ from this modular space into $L_p(\Omega, P)$. In addition, under certain conditions on Λ, we shall show that the inverse of this map is also continuous. We also point out that these results on stochastic integrals unify and extend the corresponding results of [**23**, **29**, **30**]; further, using these results, we show that one can easily recover, in a unified way, the results concerning stochastic integrals and the space of Λ-integrable functions obtained in [**2**, **7**, **20**, **27**];

We begin with some preliminaries. Let q be a non-negative number such that

$$\textbf{(MC)} \qquad E|\Lambda(A)|^q < \infty, \qquad \text{for all } A \epsilon S.$$

Throughout this section, we shall assume that the above condition is satisfied and $q \in [0, \infty)$ is fixed (note that every Λ satisfies **MC** with $q = 0$). Hence, using the standard fact which states that for an **ID** distribution μ with Lévy measure G, $\int_{\mathbf{R}} |x|^q \mu(dx)$ is finite if and only if $\int_{\{|x|>1\}} |x|^q G(dx)$ is finite, we have

$$\int_A \left[\int_{\{|x|>1\}} |x|^q \rho(s, dx) \right] \lambda(ds) = \int_{\{|x|>1\}} |x|^q F_A(dx) < \infty ,$$

for every $A \in S$ (recall F_A is the Lévy measure of $\mathcal{L}(\Lambda(A))$). Hence λ-a.e.

$$\int_{\{|x|>1\}} |x|^q \rho(s, dx) < \infty . \tag{3.1}$$

Thus, without loss of generality, we may (and do) assume that (3.1) holds for all $s \in S$. Define, for $0 \leq p \leq q$, $u \in \mathbf{R}$ and $s \in S$,

$$\Phi_p(u, s) = U^*(u, s) + u^2 \sigma^2(s) + V_p(u, s) , \tag{3.2}$$

where

$$U^*(u, s) = \sup_{|c| \leq 1} |U(cu, s)|$$

and

$$V_p(u, s) = \int_{-\infty}^{\infty} \left\{ |ux|^p I(|ux| > 1) + |ux|^2 I(|ux| \leq 1) \right\} \rho(s, dx) .$$

Next we state and prove two lemmas which will be needed for the identification of the space of Λ-integrable functions as well as for the proof of the continuity of the stochastic integral mapping and its inverse.

LEMMA 3.1. *The following are satisfied:*

(i) *for every $s \in S$, $\Phi_p(\cdot, s)$ is a continuous non-decreasing function on $[0, \infty)$ with $\Phi_p(0, s) = 0$,*

(ii) $\lambda(\{s : \Phi_p(u, s) = 0$ *for some* $u = u(s) \neq 0\}) = 0$,

(iii) *there exists a numerical constant $C > 0$ such that*

$$\Phi_p(2u, s) \leq C \Phi_p(u, s) ,$$

for all $u \geq 0$ and $s \in S$.

PROOF: It is easy to prove that $U(\cdot, s)$ is continuous; using this one proves as easily that $U^*(\cdot, s)$ is also continuous. Using this fact and the Dominated Convergence Theorem, we establish the continuity of $\Phi_p(\cdot, s)$. To see that $\Phi_p(\cdot, s)$ is non-decreasing we observe that $U^*(\cdot, s)$ is non-decreasing and, for each fixed u,

$$|ux|^p I(|xu| > 1) + |xu|^2 I(|xu| \le 1) = \begin{Bmatrix} \min\{|xu|^p, |xu|^2\} & \text{if } 0 \le p \le 2 \\ \max\{|xu|^p, |xu|^2\} & \text{if } p > 2 \end{Bmatrix} \tag{3.3}$$

is increasing in $x \ge 0$. Now we prove (ii). If $\Phi_p(u, s) = 0$, for some $u = u(s) \ne 0$, then $\rho(s, \mathbf{R}) = 0$, $\sigma^2(s) = 0$ and $U(u, s) = 0$. By the definition of $U(u, s)$, we get $a(s) = 0$. Therefore,

$$\begin{aligned} S_0 &\equiv \{s : \Phi_p(u, s) = 0 \text{ for some } u = u(s) \ne 0\} \\ &= \{s : a(s) = \sigma^2(s) = \rho(s, \mathbf{R}) = 0\}. \end{aligned}$$

(Note that above equality also establishes the measurability of S_0). Let A be any measurable subset of S_0. Since $\nu_0(A) = \int_A a(s)\lambda(ds) = 0$, we get $|\nu_0|(S_0) = 0$. Thus

$$\lambda(S_0) = |\nu_0|(S_o) + \int_{S_0} \sigma^2(s)\lambda(ds) + \int_{S_0} \min\{1, |x|^2\}\rho(s, dx) = 0.$$

To prove (iii), we use Lemma 2.8 and (3.3), and get

$$\begin{aligned} \Phi_p(2u, s) &\le 2|U(u, s)| + 27V_0(u, s) + 4u^2\sigma^2(s) + (2^p + 4)V_p(u, s) \\ &\le (2^p + 31)\Phi_p(u, s). \end{aligned}$$

LEMMA 3.2. *Let $\{\mu_n\}$ be a sequence of* **ID**. *prob. measures on* $\mathbf{R}$ *with Lévy representation:* $\mu_n \equiv (a_n, \sigma_n^2, G_n)$. *Assume* $\mu_n \xrightarrow{\omega} \delta_0$; *equivalently,* $a_n \to 0$, $\sigma_n^2 \to 0$ *and* $\int_{-\infty}^{\infty} \min\{1, |x|^2\}\, dG_n \to 0$. *Then, for any* $b > 0$,

$$\int_{\mathbf{R}} |x|^b \mu_n(dx) \to 0 \Longleftrightarrow \int_{\{|x|>1\}} |x|^b G_n(dx) \to 0.$$

(It is, of course, assumed here that $\int_R |x|^b d\mu_n < \infty$ *(and hence* $\int_{\{|x|>1\}} |x|^b G_n(dx) < \infty$*), for all* n*).*

PROOF: Under the hypothese of the Lemma, it is easy to prove that

$$\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b G_n(dx) = 0 \Longleftrightarrow \lim_n \int_{\{|x|>1\}} |x|^b G_n(dx) = 0, \tag{3.4}$$

and

$$\text{(3.5)} \qquad \lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b \mu_n(dx) = 0 \Longleftrightarrow \lim_n \int_{\{|x|>1\}} |x|^b \mu_n(x) = 0.$$

Now assume $\int_{\{|x|>1\}} |x|^b G_n(dx) \to 0$, hence, by (3.4) and Theorem 2 of [**10**], (note that $\{\mu_n\}$ is compact) $\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b \mu_n(dx) = 0$. Thus, by (3.5), $\int_{\{|x|>1\}} |x|^b \mu_n(dx) \to 0$. But, as $\mu_n \xrightarrow{\omega} \delta_0$, we have $\int_{\{|x|\le 1\}} |x|^b \mu_n(dx) \to 0$. This proves $\int_{\mathbf{R}} |x|^b \mu_n(dx) \to 0$. Conversely, if $\int_{\mathbf{R}} |x|^b \mu_n(dx) \to 0$, then, by (3.5), $\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b \mu_n(dx) = 0$. Thus by [**10**] again, $\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b G_n(dx) = 0$; which along with (3.4) imply that $\lim_n \int_{\{|x|>1\}} |x|^b G_n(dx) = 0$. ∎

In order to get ready to state and prove our first main result of this section, we will need a few more notations and definitions:

We define the so-called Musielak-Orlicz space

$$L_{\Phi_p}(S;\lambda) = \left\{ f \epsilon L_0(S;\lambda) : \int_S \Phi_p(|f(s)|, s)\, \lambda(ds) < \infty \right\}.$$

The following properties of $L_{\Phi_p}(S;\lambda)$ (which are well-known for general Musielak-Orlicz spaces generated by functions which satisfy (i), (ii) and (iii) of Lemma 3.3) will be used throughout this paper: The space $L_{\Phi_p}(S;\lambda)$ is a complete linear metric space with the F-norm defined by

$$\|f\|_{\Phi_p} = \inf\left\{ c > 0 : \int_S \Phi_p(c^{-1}|f(s)|, s)\lambda(ds) \le c \right\}.$$

Simple functions are dense in $L_{\Phi_p}(S;\lambda)$ and the natural embedding of $L_{\Phi_p}(S;\lambda)$ into $L_0(S;\lambda)$ is continuous (here $L_0(S;\lambda)$ is equipped with the topology of convergence in λ measure on every set of finite λ-measure). Finally, $\|f_n\|_{\Phi_p} \to 0$ if and only if $\int_S \Phi_p(|f_n(s)|, s)$ $\lambda(ds) \to 0$. For these and further facts concerning Musielak-Orlicz spaces, we refer the reader to [**16**].

THEOREM 3.3. *Let $0 \le p \le q$ and Φ_p be as in (3.2). Then*

$$\left\{ f : f \text{ is } \Lambda\text{-integrable and } E\left|\int_S f d\Lambda\right|^p < \infty \right\} = L_{\Phi_p}(S;\lambda),$$

and the linear mapping

$$L_{\Phi_p}(S;\lambda) \ni f \longmapsto \int_S f d\Lambda \in L_p(\Omega; P)$$

is continuous (note that $p = 0$ here signifies that $L_{\Phi_0}(S;\lambda) = \{f : f$ denotes Λ-integrable$\}$).

PROOF: Let $f \in L_{\Phi_p}(S;\lambda)$; i.e. $\int_S \Phi_p(|f(s)|, s)\,\lambda(ds) < \infty$. Then, it is easy to see that the conditions (i), (ii) and (iii) of Theorem 2.7 are satisfied, so, f is Λ-integrable. If F_f denotes the Lévy measure of $\mathcal{L}(\int_S f\,d\Lambda)$ (see Theorem 2.7), then we have

$$\begin{aligned}(3.6)\qquad \int_{|u|>1} |u|^p F_f(du) &= \int_S \left[\int_{\{|f(s)x|>1\}} |f(s)x|^p \rho(s,dx)\right] \lambda(ds)\\ &\le \int_S \Phi_p(|f(s)|, s)\,\lambda(ds) < \infty;\end{aligned}$$

and, consequently, $E\left|\int_S f\,d\Lambda\right|^p < \infty$.

Conversely, assume that f is Λ-integrable and $E\left|\int_S f\,d\Lambda\right|^p < \infty$. By Lemma 2.8 and (i) and (iii) of Theorem 2.7, we get

$$\int_S U^*(|f(s)|, s)\,\lambda(ds) \le \int_S \left|U(f(s), s)\right| \lambda(ds) + 8\int_S V_0(f(s), s)\lambda(ds) < \infty.$$

Since $E\left|\int_S f\,d\Lambda\right|^p < \infty$, we have $\int_{\{|u|>1\}} |x|^p F_f(dx) < \infty$; hence, by (3.6) and (iii) of Theorem 2.7, we get

$$\int_S V_p(f(s), s)\lambda(ds) \le \int_{\{|u|>1\}} |x|^p F_f(dx) + \int_S V_0(f(s), s)\,\lambda(ds) < \infty.$$

Combining the above and (ii) of Theorem 2.7, we get $f \in L_{\Phi_p}(S;\lambda)$.

Let $f_n \to 0$ in $L_{\Phi_p}(S;\lambda)$; i.e.

$$(3.7)\qquad \int_S \Phi_p(|f_n(s)|, s)\,\lambda(ds) \to 0 \text{ as } n \to \infty.$$

Let a_n, σ_n^2 and F_n be, respectively, the centering constant, the variance and the Lévy measure in the canonical representation of the ch. function of $\mathcal{L}\left(\int_S f_n d\Lambda\right)$ (see (iv) of Theorem 2.7). Then (3.7) implies that $a_n \to 0$, $\sigma_n^2 \to 0$ and

$$\int_{\mathbf{R}} \left\{|x|^p I(|x| > 1) + x^2 I(|x| \le 1)\right\} F_n(dx) \to 0,$$

as $n \to \infty$. Thus, in view of Lemma 3.2, $E\left|\int_S f_n d\Lambda\right|^p \to 0$, as $n \to \infty$ if $p > 0$; and, if $p = 0$, then clearly $\int_S f_n d\Lambda \to 0$ in prob. ∎

We shall study now the conditions under which the mapping $f \to \int_S f\,d\Lambda$ is an isomorphism. First we note that, in general, this mapping is not one-to-one. Indeed, if $\Lambda(ds) = ds$

is the (deterministic) Lebesque measure on $S = [0,1]$, then obviously $f \longmapsto \int_0^1 f(s)ds$ is not one-to-one. In view of this, one needs to impose some suitable condition on Λ (or on some of its parameters) which, on one hand, alleviates this difficulty and makes the mapping an isomorphism but, at the same time, is weak enough so that it is satisfied by a large class of **ID** r. measures. We found the following condition quite satisfactory with regard to these criterions; we refer this as **(IC)** (**I** for isomorphism) condition:

$$
\textbf{(IC)} \qquad \left\{ \begin{array}{l} \text{There exists a constant } C = C(p,q),\ 0 \le p \le q, \\ \text{such that for every } u \ge 0 \\ |U(u,s)| \le C\left\{u^2\sigma^2(s) + V_p(u,s)\right\} \qquad \text{a.e. } [\lambda]. \end{array} \right\}
$$

The following is our second main result of this section.

THEOREM 3.4. *Let* (**IC**) *be satisfied for some* $0 \le p \le q$. *Then the mapping* $f \to \int_S f d\Lambda$ *is an isomorphism from* $L_{\Phi_p}(S;\lambda)$ *into* $L_p(\Omega;P)$. *Moreover,*

$$
\left\{ \int_S f d\Lambda : f \in L_{\Phi_p}(S;\lambda) \right\} = \overline{\text{lin}}\left\{\Lambda(A) : A \in S\right\}_{L_p(\Omega;P)} .
$$

PROOF: By Lemma 2.8 and (**IC**), we get, for every $u \ge 0$,

$$
\begin{aligned}
U^*(u,s) &\le |U(u,s)| + 8V_0(u,s) \\
&\le C_1\left\{u^2\sigma^2(s) + V_p(u,s)\right\}
\end{aligned}
\tag{3.8}
$$

a.e. $[\lambda]$, where $C_1 \le C + 8$.

Let $E\left|\int_S f_n d\Lambda\right|^p \to 0$, if $p > 0$ or $\int_S f_n d\Lambda \to 0$ in prob. if $p = 0$. By Theorem 2.7 (iv) and Lemma 3.2, we have

$$
\int_S |f_n(s)|^2 \sigma^2(s)\lambda(ds) = \sigma^2_{f_n} \to 0
$$

and

$$
\int_S V_p\left(f_n(s), s\right)\lambda(ds) = \int_{\mathbf{R}} \left\{|x|^p I\left(|x| > 1\right) + |x|^2 I\left(|x| \le 1\right)\right\} F_{f_n}(dx) \to 0,
$$

as $n \to \infty$, where $\sigma^2_{f_n}$ and F_{f_n} are respectively, the variance and the Lévy measure in the canonical representation of the ch. function of $\mathcal{L}\left(\int_S f_n d\Lambda\right)$. Thus, by (3.8), we have

$$
\int_S U^*\left(|f_n(s)|, s\right)\lambda(ds) \to 0 \text{ as } n \to \infty.
$$

Therefore, $\int_S \Phi_p(|f_n(s)|, s)\, \lambda(ds) \to 0$; i.e., $f_n \longrightarrow 0$ in $L_{\Phi_p}(S;\lambda)$. This proves the invertability of the map $f \longmapsto \int_S f d\Lambda$ and the continuity of the inverse map.

Using the fact that simple functions are dense in $L_{\Phi_p}(S;\lambda)$ and that

$$\operatorname{lin}\{\Lambda(A) : A \in \mathcal{S}\} = \left\{ \int_S f d\Lambda : f \text{ is simple} \right\},$$

the proof of the last statement of the theorem is easy. ∎

Corollary 3.5. Let **(IC)** be satisfied for some $0 \le p \le q$ and $\int_S f_n d\Lambda \to 0$ in $L_p(\Omega; P)$. Then $f_n \to 0$ in λ on any set of λ-finite measure.

PROOF: It follows from Theorem 3.4 and the earlier noted fact that the natural embedding of L_{Φ_p} into $L_0(S;\lambda)$ is continuous. ∎

The **(IC)** condition is imposed on certain parameters of Λ and not directly on Λ; this limits the usefulness of Theorem 3.4 somewhat. Thus, it is desirable to find sufficient conditions directly in terms of Λ which guarantee **(IC)** and hence also the fact that the integral mapping is an isomorphism. We shall provide such sufficient conditions in Propositions 3.6 and 3.8.

PROPOSITION 3.6. *The condition* **(IC)** *is satisfied under any of the following two hypotheses on the* **ID** *r. measure* Λ *and the real number* p*:*

(i) Λ *is symmetric and* $0 \le p \le q$ *arbitrary,*
(ii) $E[\Lambda(A)] = 0$ *for all* A *and* $1 \le p \le q$.

PROOF: That **(IC)** holds under (i) is trivial, since in this case $a(s) = 0$ and $\rho(s,\cdot)$ is symmetric, which implies that $U(\cdot, s) \equiv 0$ a.e. $[\lambda]$. Now we prove that **(IC)** holds under (ii). Since $E|\Lambda(A)|^q < \infty$, $q \ge 1$ and $E\{\Lambda(A)\} = 0$, we have

$$\begin{aligned} \hat{\mathcal{L}}(\Lambda(A))(t) &= \exp\left\{ -\frac{1}{2} t^2 \nu_1(A) + \int_{\mathbf{R}} \left(e^{itx} - 1 - itx\right) F_A(dx) \right\} \\ &= \exp\left\{ it\nu_0(A) - \frac{1}{2} t^2 \nu_1(A) + \int_{\mathbf{R}} \left(e^{itx} - 1 - it\tau(x)\right) F_A(dx) \right\}, \end{aligned} \tag{3.9}$$

where $\nu_0(A) = \int_{\mathbf{R}} [\tau(x) - x]\, F_A(dx)$. Hence, by Proposition 2.5, a.e. $[\lambda]$,

$$a(s) = \int_{\mathbf{R}} (\tau(x) - x)\, \rho(s, dx) \text{ and } U(u,s) = \int_{\mathbf{R}} (\tau(ux) - ux)\, \rho(s, dx). \tag{3.10}$$

Thus we get, for every $p \geq 1$,

$$|U(u,s)| \leq \int_{\{|ux|>1\}} |\tau(ux) - ux|\,\rho(s,dx)$$
$$\leq \int_{\{|ux|>1\}} |ux|\rho(s,dx) \leq V_p(u,s)$$

a.e. $[\lambda]$, which concludes the proof. ∎

As we noted in Section II, our definition of stochastic integrals is the same as advocated first by Urbanik and Woyczynski [**30**] and Urbanik [**29**] and later adopted by Rosinski [**23**]. Thus our results on stochastic integrals of *real* functions relative to arbitrary **ID** r. measures do unify and extend the pertinent results of these authors. Another approach of defining stochastic integrals relative to symmetric $\mathbf{S}(\alpha)$, and symmetric $\mathbf{S}(r,\alpha)$ and centered $\mathbf{S}(r,\alpha)$, r. measures Λ have been taken in [**2, 27**] and [**20**], respectively. In these papers, the integral $\int f d\Lambda$ is defined as L_p-limit, $0 < p < \alpha$, of a sequence of integrals of simple functions relative to Λ; and it is shown that the space of Λ-integrable functions is the $L_\alpha(\lambda)$-space and that the integral map $L_\alpha(\lambda) \ni f \longmapsto \int f d\Lambda \in L_p(P)$ is a topological and linear isomorphism. The rest of this section is devoted to show that our integrals as well as the space L_{Φ_p} of Λ-integrable function do coincide with those of [**2, 27**] and [**20**], when Λ is symmetric $\mathbf{S}(\alpha)$, and symmetric $\mathbf{S}(r,\alpha)$ or centered $\mathbf{S}(r,\alpha)$ r. measures, respectively; and, that the integral map satisfies the above cited property. Thus, we recover all these results of [**2**, **27**, **20**] in a unified way. Finally, towards the end of this section we point out certain facts about Λ-integrable functions for certain $\mathbf{S}(r,1)$ r. measures.

If Λ is a centered $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) r. measure where $1 < \alpha < 2$, then $E\,|\Lambda(A)|^q < \infty$, for any $q < \alpha$, and $E\Lambda(A) = 0$, for every $A \in \mathcal{S}$. Hence the ch. function of $\Lambda(A)$ is of the form (3.9), where $\nu_1 \equiv 0$ and F_A is an $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) Lévy measure.

If Λ is a centered $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) r. measure and $0 < \alpha < 1$, then

$$\text{(3.11)} \qquad \hat{\mathcal{L}}\left(\Lambda(A)\right)(t) = \exp\left\{\int_R \left(e^{itx} - 1\right) F_A(dx)\right\}$$
$$= \exp\left\{it\nu_0(A) + \int_R \left(e^{itx} - 1 - it\tau(x)\right) F_A(dx)\right\},$$

where $\nu_0(A) = \int_{\mathbf{R}} \tau(x)F_A(dx)$ and F_A is an $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) Lévy measure for every $A \in \mathcal{S}$. Therefore, we have (see Proposition 2.5 and Theorem 2.7)

$$\text{(3.12)} \qquad a(s) = \int_{\mathbf{R}} \tau(x)\rho(s,dx) \text{ and } U(u,s) = \int_{\mathbf{R}} \tau(ux)\rho(s,dx) \qquad \text{a.e. } [\lambda].$$

Finally, if Λ is a centered $\mathbf{S}(1)$ (resp. $\mathbf{S}(r,1)$) r. measure, then Λ is symmetric and the ch. function of $\Lambda(A)$ is given by (2.1) with $\nu_0 \equiv \nu_1 \equiv 0$ and F_A being a symmetric $\mathbf{S}(1)$ (resp. $\mathbf{S}(r,1)$) r. measure, for every $A \in \mathcal{S}$.

In the following lemma, we state the fact that the conditional Lévy measures $\rho(s,\cdot)$ of $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) r. measure Λ are $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$). The proof of this fact is postponed to the next section mainly for convenience but also because this fact has more relevance there. Formula (3.15) below follow from (3.14) by a standard argument. The proof of (3.14) can be found in [**20**].

LEMMA 3.7. *(a). Let Λ be a $\mathbf{S}(\alpha)$ r. measure. Then a.e. $[\lambda]$*

$$\rho(s,dx) = c_1(s)I(x>0)x^{-1-\alpha}dx + c_{-1}(s)I(x<0)|x|^{-1-\alpha}dx, \tag{3.13}$$

where $c_1, c_{-1} : S \longmapsto [0,\infty)$ are $\sigma(S) - \mathcal{B}\,[0,\infty)$ measurable.

(b) Let Λ be a $\mathbf{S}(r,\alpha)$ r. measure. Then, for λ almost all $s \in S$,

$$\rho(s,B) = \sum_{n=-\infty}^{\infty} r^n \rho\left(s, (r^{\frac{n}{\alpha}}B)\cap\Delta\right) \text{ for all } B \in \mathcal{B}(\mathbf{R}), \tag{3.14}$$

where $\Delta = \left\{x \in \mathbf{R} : r^{\frac{1}{\alpha}} < |x| \le 1\right\}$. More generally, for λ-almost all $s \in S$, the following formulas hold

$$\begin{cases} \int_{\mathbf{R}} f(x)\rho(s,dx) & = \sum_{n=-\infty}^{\infty} r^n \int_\Delta f(r^{\frac{-n}{\alpha}}x)\rho(s,dx), \\ \int_{|x|>r^{\frac{k}{\alpha}}} f(x)\rho(s,dx) & = \sum_{i=1}^{\infty} r^{-k+i} \int_\Delta f\left(r^{\frac{k-i}{\alpha}}x\right)\rho(s,dx), \\ \int_{|x|\le r^{\frac{k}{\alpha}}} f(x)\rho(s,dx) & = \sum_{i=o}^{\infty} r^{-k-i} \int_\Delta f\left(r^{\frac{k+i}{\alpha}}x\right)\rho(s,dx), \end{cases} \tag{3.15}$$

for every Borel non-negative function f and an arbitrary integer k.

PROPOSITION 3.8. *Let Λ be a centered $\mathbf{S}(\alpha)$, or more generally, a centered $\mathbf{S}(r,\alpha)$ r. measure. Then the* (**IC**) *condition holds, for any $0 \le p < \alpha$, and $L_{\Phi_p}(S;\lambda) = L_\alpha(S;\lambda)$ up to a renorming, for every $0 \le p < \alpha$. Consequently, there are positive constants C_1 and C_2 depending only on p, r and α such that*

$$C_1\left(\int_S |f|^\alpha d\lambda\right)^{\frac{1}{\alpha}} \le \left(E\left|\int_S f\,d\Lambda\right|^p\right)^{\frac{1}{p}} \le C_2\left(\int_S |f|^\alpha d\lambda\right)^{\frac{1}{\alpha}}, \tag{3.16}$$

for every $f \in L_\alpha(S;\lambda)$.

PROOF: Since every centered $\mathbf{S}(\alpha)$ r. variable is also a centered $\mathbf{S}(r,\alpha)$ r. variable for every $0 < r < 1$, it is enough to prove the proposition for the case when Λ is a centered $\mathbf{S}(r,\alpha)$ r. measure.

First we shall bound $U(u,s)$. If $0 < \alpha < 1$, then by (3.12), we have

$$(3.17) \quad |U(u,s)| \le \int_{\mathbf{R}} |\tau(ux)|\,\rho(s,dx) = |u| \int_{\{|x| \le |u|^{-1}\}} |x|\rho(s,dx) + \int_{\{|x|>|u|^{-1}\}} \rho(s,dx)$$

(for the sake of brevity we shall omit in this proof the phrase "for λ-almost all s"). Let k be an integer such that $r^{\frac{k}{\alpha}} < |u|^{-1} \le r^{\frac{k-1}{\alpha}}$. Using (3.15), we obtain

$$\begin{aligned}\int_{\{|x|\le|u|^{-1}\}} |x|\rho(s,dx) &\le \int_{\{|x|\le r^{\frac{k-1}{\alpha}}\}} |x|\rho(s,dx)\\ &= \sum_{i=0}^{\infty} r^{-k+1-i} \int_{\Delta} r^{\frac{k-1+i}{\alpha}} |x|\rho(s,dx)\\ &\le \sum_{i=0}^{\infty} r^{(\frac{1}{\alpha}-1)(k-1+i)}\rho(s,\Delta)\\ &\le \frac{r^{1-\frac{1}{\alpha}}}{1-r^{\frac{1}{\alpha}-1}}\rho(s,\Delta)|u|^{\alpha-1};\end{aligned}$$

and, again by (3.15), we get

$$\begin{aligned}\int_{\{|x|>|u|^{-1}\}} \rho(s,dx) &\le \int_{\{|x|>r^{\frac{k}{\alpha}}\}} \rho(s,dx)\\ &= \sum_{i=1}^{\infty} r^{-k+i} \int_{\Delta} \rho(s,dx) \le \frac{1}{1-r}\rho(s,\Delta)|u|^{\alpha}.\end{aligned}$$

By combining the above and (3.17), we obtain

$$(3.18) \qquad |U(u,s)| \le D\rho(s,\Delta)|u|^{\alpha},$$

where $D = r^{1-\frac{1}{\alpha}}\left(1-r^{\frac{1}{\alpha}-1}\right)^{-1} + (1-r)^{-1}$. Let now $1 < \alpha < 2$. Then, by (3.10), we get

$$(3.19) \qquad |U(u,s)| \le \int_{\{|xu|>1\}} |\tau(xu) - xu|\,\rho(s,dx) \le |u| \int_{\{|x|>|u|^{-1}\}} |x|\rho(s,dx).$$

Let k be as above. Utilizing (3.15) again, we obtain

$$\begin{aligned}\int_{\{|x|>|u|^{-1}\}} |x|\rho(s,dx) &\le \int_{\{|x|>r^{\frac{k}{\alpha}}\}} |x|\rho(s,dx)\\ &= \sum_{i=1}^{\infty} r^{-k+i} \int_{\Delta} r^{\frac{k-i}{\alpha}} |x|\rho(s,dx)\\ &\le \sum_{i=1}^{\infty} r^{(1-\frac{1}{\alpha})(i-k)}\rho(s,\Delta)\\ &\le \left(1-r^{1-\frac{1}{\alpha}}\right)^{-1}\rho(s,\Delta)|u|^{\alpha-1},\end{aligned}$$

which, together with (3.19), shows that (3.18) holds for all $1 < \alpha < 2$ with $D = \left(1 - r^{1-\frac{1}{\alpha}}\right)^{-1}$.

Using (3.15) repeatedly, in a very similar way as above, one can find positive constants D_1 and D_2, depending only on p, r and α, where $0 \le p < \alpha$, $0 < r < 1$ and $0 < \alpha < 2$, such that

(3.20)
$$\begin{aligned} D_1\rho(s,\Delta)|u|^\alpha &\le V_p(u,s) \\ &= u^2\int_{\{|xu|\le 1\}} x^2\rho(s,dx) + |u|^p\int_{\{|xu|>1\}} |x|^p\rho(s,dx) \\ &\le D_2\rho(s,\Delta)|u|^\alpha. \end{aligned}$$

The condition (**IC**) follows now by (3.18) and (3.20) since, if $\alpha \ne 1$,

$$|U(u,s)| \le D\rho(s,\Delta)|u|^\alpha \le DD_1^{-1}V_p(u,s).$$

If $\alpha = 1$, $\rho(s,\cdot)$ is symmetric and $a(s) = 0$; which implies $U(\cdot,s) \equiv 0$ and (**IC**) holds in this case trivially.

Combining (3.18) and (3.20) we get, for every $0 \le p < \alpha$ and $0 < \alpha < 2$ (including $\alpha = 1$),

$$\begin{aligned} D_1\rho(s,\Delta)|u|^\alpha &\le \Phi_p(u,s) = U^*(u,s) + V_p(u,s) \\ &\le (D + D_2)\rho(s,\Delta)|u|^\alpha, \end{aligned} \tag{3.21}$$

where $D = 0$, if $\alpha = 1$. We shall obtain now bounds for $\rho(s,\Delta)$ utilizing (2.5); which, in view of (3.10) and (3.12), reads

$$|U(1,s)| + V_0(1,s) = 1, \qquad \text{if } \alpha \ne 1,$$

and $V_0(1,s) = 1$, if $\alpha = 1$. By (3.18) and (3.20), we get

$$D_1\rho(s,\Delta) \le |U(1,s)| + V_0(1,s) = 1 \le (D + D_2)\rho(s,\Delta);$$

hence

$$(D + D_2)^{-1} \le \rho(s,\Delta) \le D_1^{-1}.$$

Consequently, by (3.21),

$$D_1(D + D_2)^{-1}|u|^\alpha \le \Phi_p(u,s) \le D_1^{-1}(D + D_2)|u|^\alpha.$$

This shows that $f \in L_{\Phi_p}$ if and only if $\|f\|_\alpha^\alpha = \int_S |f|^\alpha d\lambda < \infty$ and obviously the F-norms $\|\cdot\|_{\Phi_p}$ and $\|\cdot\|_\alpha^{\min\{1,\alpha\}}$ are comparable. Now, the inequalities (3.16) follow from Theorem 3.4 and the Closed Graph Theorem. ∎

Remark 3.9. If Λ is a $\mathbf{S}(\alpha)$ (resp. a $\mathbf{S}(r, \alpha)$) r. measure and $\alpha \neq 1$, then one can find a (non-random) signed measure, say, λ_1 such that $\Lambda_1 = \Lambda - \lambda_1$ is a centered $\mathbf{S}(\alpha)$ (resp. a centered $\mathbf{S}(r, \alpha)$) r. measure (see [**20**] for a similar decomposition of $\mathbf{S}(r, \alpha)$ stochastic process). Thus the stochastic integral with respect to Λ is equal to the sum of the stochastic integral with respect to Λ_1, whose properties have been described in the previous proposition, and the usual Lebesque integral with respect to λ_1. Such a decomposition is not possible if $\alpha = 1$. However, if Λ is an arbitrary $\mathbf{S}(r, 1)$ r. measure with $\nu_0 \equiv 0$, then, using (3.15), one can obtain the following bound:

$$|U(u, s)| \leq A\rho(s, \Delta)|u| \left(1 + |(\log |u|)|\right),$$

where $A = \max\left\{2(1-r)^{-1},\ |\log r|^{-1}\right\}$. By (2.5) we have $V_0(1, s) = 1$, which implies, by (3.20) (which holds in the non-centered case with the same constants),

$$D_2^{-1} \leq \rho(s, \Delta) \leq D_1^{-1}.$$

Thus

$$|U(u, s)| \leq AD_1^{-1}|u| \left(1 + |(\log |u|)|\right).$$

Since $u \to u\,(1 + |\log u|)$ is increasing on $\mathbf{R}_+$, we get

$$U^*(u, s) \leq AD_1^{-1} u\,(1 + |\log u|), \qquad u > 0.$$

Finally, by (3.20) and the above inequality,

$$A_1 u \leq \Phi_p(u, s) \leq A_2 u\,(1 + |\log u|),\ u > 0,$$

where A_1 and A_2 depend only on r and $p < 1$. We conclude that

$$L \log L(S; \lambda) \subseteq L_{\Phi_p}(S; \lambda) \subseteq L_1(S; \lambda),$$

where $L \log L(S; \lambda)$ is the Orlicz space based on $\Phi(u) = u\,(1 + |\log u|)$. This generalizes a result in [**7**], proven for the $\mathbf{S}(1)$-case.

IV. Spectral Representations of General Discrete Parameter ID Processes

Let M be a $\mathbf{S}(\alpha)$ Lévy measure on $\ell_2 = \ell_2(N)$; then, as is well known [**13**], M admits the representation:

$$M = (\rho_0 \times \gamma_0) \circ \Psi_0^{-1}, \tag{4.1}$$

where γ_0 is a *finite* measure on ∂U, the boundary of the unit ball in ℓ_2, ρ_0 is a $S(\alpha)$ Lévy measure on R and Ψ_0 is the map: $\partial U \times R^+ \longrightarrow \ell_2 \setminus \{0\}$ defined by $\Psi_0(u, x) = xu$. It is noted in [**20**, **21**] that a representation similar to (4.1), can be obtained for any $S(r, \alpha)$ Lévy measure but one must replace ∂U by the annulus $\Delta = \{x : r^{\frac{1}{\alpha}} < \|x\| \leq 1\}$. This fact that M admits the representation like (4.1) plays perhaps the most crucial role in the proofs of spectral representations of stable and semistable processes obtained in [**2**, **7**, **8**, **13**, **20**, **21**, **27**, **28**]. The basic idea of all these proofs is as follows: Given a stable (resp. semistable) process $X = \{X_n\}$ with paths in ℓ_2, one first represents the Lévy measure M of $\mathcal{L}(X)$ as in (4.1), then one defines a r. measure Λ on ∂U (resp. on Δ) (or via a Borel isomorphism on some other Borel subset of a complete separable metric space) with control measure

$$F_A(B) = \gamma_0(A)\rho_0(B);$$

and, finally by choosing suitable functions f_n, one shows that

$$\left\{\int_S f_n d\Lambda\right\} \stackrel{d}{=} \{X_n\}. \tag{4.2}$$

Further, using some continuity arguments, one obtains representation like (4.2) for continuous parameter stable and semistable processes.

In order to apply a similar approach to obtain spectral representations of general **ID** processes, it is thus necessary to obtain a suitable representation, similar to (4.1), for an arbitrary Lévy measure M on ℓ_2. In order to make sure that the r. measure Λ retains properties similar to those of the given **ID** process X, it is important that this representation be such that both $F_A(\cdot)$ and ρ_0 inherit properties of M (the Lévy measure of $\mathcal{L}(X)$); e.g., if M belongs to a known class (say stable, semistable or self-decomposable) of Lévy measures on ℓ_2 then $F_A(\cdot)$ and ρ_0 belong to the same class of Lévy measures on R. That such a representation is possible is shown in Theorem 4.2. As we will see, this representation helps us obtain spectral representations of most **ID** processes in a unified way which include and extend, to a large degree, all known spectral representations to date of various special **ID** processes.

We begin by introducing some notations and conventions which will remain fixed throughout *this* section and the *next*, unless explicitly stated otherwise.

The notations ∂U and Ψ_0 are as above: $\partial U = \{z \epsilon \ell_2 : \|z\| = 1\}$; $\Psi_0 : \partial U \times R^+ \longrightarrow \ell_2 \setminus \{0\}$ is the Borel isomorphism, defined by, $\Psi_0(u, x) = xu$ (note $\Psi_0^{-1}(z) = \left(\frac{z}{\|z\|}, \|z\|\right)$); the natural (Borel-measurable extension of Ψ_0 to $\partial U \times R_0$, we denote, by $\overline{\Psi}_0$, where

$R_0 = R \setminus \{0\}$ and, as noted before $R^+ = (0,\infty)$). Next we denote, by S, an arbitrary uncountable Borel subset of a complete separable metric space, by φ, a Borel isomorphism from ∂U onto S, (see Theorems 2.12 and 2.8 of [17]) and, by I_φ and Ψ, respectively, the Borel isomorphisms from $\partial U \times R^+$ onto $S \times R^+$ and from $S \times R^+$ onto $\ell_2 \setminus \{0\}$, which are defined by

$$I_\varphi(u,x) = (\varphi(u), x) \text{ and } \Psi(s,x) = x\varphi^{-1}(s).$$

Finally, we denote, by $\bar{I}_\varphi$, the natural extension of I_φ to $\partial U \times R_0$, and, by $\overline{\Psi}$, the natural extension of Ψ to $S \times R_0$. (Clearly, $\bar{I}_\varphi$ is a Borel isomorphism onto $S \times R_0$ and $\overline{\Psi}$ is a Borel measurable map onto $\ell_2 \setminus \{0\}$). To keep easy track of these maps, spaces and the measures (to be defined in the following), we found the following (commutative) diagrams useful:

$$\begin{tikzcd}
\ell_2 \setminus \{\theta\} \arrow[r, "\Psi_0"] & \partial U \times R^+ \arrow[d, shift right, "I_\varphi"'] \\
 & S \times R^+ \arrow[u, shift right, "I_\varphi^{-1}"'] \arrow[ul, "\Psi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\ell_2 \setminus \{\theta\} \arrow[r, "\overline{\Psi}_0"] & \partial U \times R_0 \arrow[d, shift right, "\bar{I}_\varphi"'] \\
 & S \times R_0 \arrow[u, shift right, "\bar{I}_\varphi^{-1}"'] \arrow[ul, "\overline{\Psi}"]
\end{tikzcd}$$

Diagram 4.1

Now we shall define certain measures related to a given Lévy measure on ℓ_2; notations, used for these measures, will also remain fixed throughout this section.

Given a Lévy measure M on ℓ_2, the finite measure Γ_0 on $\mathcal{B}(\partial U \times R^+)$, defined by,

$$\Gamma_0 = M_0 \circ \Psi_0, \qquad \text{where } M_0(dz) = \min(1, \|z\|^2)\, M(dz), \tag{4.3}$$

can be represented, by Proposition 2.4, as

$$\Gamma_0(C) = \int_{\partial U} \left(\int_{R^+} I_C(u,x) q(u,dx) \right) \gamma_0(du), \tag{4.4}$$

where $q : \partial U \times \mathcal{B}(R^+) \longmapsto [0,1]$ satisfies conditions analogous to (d) and (e) of Proposition 2.4, and γ_0 is the finite measure defined by

$$\gamma_0(A) \equiv \Gamma_0(A \times R^+) = \int_{\{z : \frac{z}{\|z\|} \in A\}} \min(1, \|z\|^2) M(dz), \tag{4.5}$$

for every $A \in \mathcal{B}(\partial U)$. Now, we define the measures γ on $\mathcal{B}(S)$, $\rho(s,\cdot)$ on $\mathcal{B}(R^+)$ and F on $\mathcal{B}(S \times R^+)$ by

$$\gamma = \gamma_0 \circ \varphi^{-1}, \qquad \rho(s,dx) = \left[\min(1,|x|^2)\right]^{-1} q\left(\varphi^{-1}(s), dx\right), \tag{4.6}$$

for every $s \epsilon S$, and

$$F(C) = \int_S \left(\int_{R^+} I_C(s,x)\rho(s,dx) \right) \gamma(ds), \tag{4.7}$$

for every $C \epsilon \mathcal{B}(S \times R^+)$. If M is symmetric, then $\Gamma_0(A \times B) = \Gamma_0(-A \times B)$, hence, in particular, γ_0 is symmetric (and we denote it by $\overline{\gamma}_0$). Using these and (4.4), we can and do assume that $q(\cdot, dx)$ chosen above is such that, for all $u \epsilon \partial U$,

$$q(u, dx) = q(-u, dx).$$

Consequently, if M is symmetric, the measures $\rho(s, \cdot)$ satisfy:

$$\rho(\varphi(u), dx) = \rho(\varphi(-u), dx), \tag{4.8}$$

for all $u \epsilon \partial U$. In addition to the measures $\rho(s, \cdot)$, γ and F, in the symmetric case, we also associate (to M) the measures $\overline{\rho}(s, \cdot)$ on $\mathcal{B}(R_0)$ and $\overline{F}$ on $\mathcal{B}(S \times R_0)$, which are defined by the formulas:

$$\overline{\rho}(s, dx) = \tfrac{1}{2} \left[\rho(s, dx) + (-1) \cdot \rho(s, dx) \right], \tag{4.9}$$

for all $s \epsilon S$, and

$$\overline{F}(C) = \int_S \left(\int_{R_0} I_C(s,x)\overline{\rho}(s,dx) \right) \overline{\gamma}(ds), \tag{4.10}$$

for all $C \epsilon \mathcal{B}(S \times R_0)$, where (for reasons to keep similar notations in the symmetric case) we write $\overline{\gamma}$ for γ. (As we noted in Section I, we will assume that $\rho(s, \cdot)$ are naturally extended to R_0 (or to R) and we will use the same notations for the extended measures. Similar remark applies to the measures $\overline{\rho}(s, \cdot)$, and to the measures $F_A(\cdot)$ and $\overline{F}_A(\cdot)$ which are defined in Lemma 4.1 (iii)).

In the following lemma, we collect a few facts about some of the measures defined above; the proofs of these facts are rather straightforward. But we record these facts here for clarity and ready reference.

LEMMA 4.1. (i) *The functions ρ and $\overline{\rho}$ satisfy analogs of (d) and (e) of Proposition 2.4.*

(ii) *The measures $\rho(s, \cdot)$ and $\overline{\rho}(s, \cdot)$ are Lévy measures on R; in fact, for all $s \epsilon S$,*

$$\int_{R^+} \min(1, |x|^2)\rho(s, dx) = \int_R \min(1, |x|^2)\overline{\rho}(s, dx) = 1; \tag{4.11}$$

further, the measures $\overline{\gamma}$ and $\overline{\rho}(s,\cdot)$ satisfy:

$$(-1)\cdot\overline{\gamma}\circ\varphi=\overline{\gamma}\circ\varphi\,(=\overline{\gamma}_0), \tag{4.12}$$

and

$$\overline{\rho}(\varphi(u),dx)=\overline{\rho}(\varphi(-u),dx) \text{ and } \overline{\rho}(s,dx)=(-1)\cdot\overline{\rho}(s,dx), \tag{4.13}$$

for all $u\epsilon\partial U$ and $s\epsilon S$.

(iii) *The measures $F_A(\cdot)\equiv F(A\times\cdot)$ and $\overline{F}_A(\cdot)\equiv\overline{F}(A\times\cdot)$ are Lévy measures on R; in fact,*

$$\int_{R^+}\min(1,x^2)F_A(dx)=\gamma(A) \quad\text{and}\quad \int_{R_0}\min(1,x^2)\overline{F}_A(dx)=\overline{\gamma}(A), \tag{4.14}$$

for all $A\epsilon\mathcal{B}(S)$; further, $\overline{F}_A(\cdot)$'s are symmetric and $\overline{F}$ satisfies (the condition of symmetry):

$$\overline{F}(\bar{I}_\varphi(C))=\overline{F}(\bar{I}_\varphi(-C)), \tag{4.15}$$

for all $C\epsilon\mathcal{B}(\partial U\times R_0)$.

(iv) *For every $C\epsilon\mathcal{B}(\partial U\times R_0)$,*

$$\overline{F}\left(\bar{I}_\varphi(C)\right)=\tfrac{1}{2}\left[F\left(I_\varphi\left(C\cap(\partial U\times R^+)\right)\right)+F\left(I_\varphi\left(-C\cap(\partial U\times R^+)\right)\right)\right] \tag{4.16}$$

(here F, $\overline{F}$ are the measures defined by (4.7) and (4.10) associated to the same symmetric Lévy measure M).

PROOF: The proof of (i) is clear by definitions; proof of (4.11) follows from the fact that $q(u,R^+)=1$, for all $u\epsilon\partial U$. The proofs of (4.12) and (4.13) follow, respectively, from (4.5), (4.6) and (4.8), (4.9). That (4.14) holds is a consequence of (4.7), (4.10) and the fact that $q(u,R^+)=1$. That $\overline{F}_A(\cdot)$ is symmetric is clear from (4.10) and (4.13); to see that (4.15) holds, we observe:

$$\begin{aligned}\overline{F}(\bar{I}_\varphi(C))&=\int_S\left(\int_{R_0}I_{\bar{I}_\varphi(C)}(s,x)\overline{\rho}(s,dx)\right)\overline{\gamma}(ds)\\&=\int_{\partial U}\left(\int_{R_0}I_{\bar{I}_\varphi(C)}\left(\varphi(u),x\right)\overline{\rho}(\varphi(u),dx)\right)\overline{\gamma}_0(du)\end{aligned}$$

(recall, from (4.12), $\overline{\gamma}=\overline{\gamma}_0\circ\varphi^{-1}$), which, by (4.12) and (4.13),

$$
\begin{aligned}
&= \int_{\partial U}\left(\int_{R_0} I_{\bar I_\varphi(C)}(\varphi(-u),-x)\bar\rho(\varphi(-u),dx)\right)\bar\gamma_0(du)\\
&= \int_{\partial U}\left(\int_{R_0} I_{\bar I_\varphi(-C)}(\varphi(u),x)\,\bar\rho\,(\varphi(u),dx)\right)\bar\gamma_0(du)\\
&= \int_{S}\left(\int_{R_0} I_{\bar I_\varphi(-C)}(s,x)\bar\rho(s,dx)\right)\bar\gamma(ds)\\
&= F(\bar I_\varphi(-C)).
\end{aligned}
$$

Finally, we prove (4.16); we observe, from (4.9) and (4.10),

$$
\begin{aligned}
\overline{F}\left(\bar I_\varphi(C)\right) &= \tfrac12\left[\int_S\left(\int_{R_0} I_{\bar I_\varphi(C)}(s,x)\rho(s,dx)\right)\bar\gamma(ds)\right.\\
&\qquad\left.+\int_S\left(\int_{R_0} I_{\bar I_\varphi(C)}(s,x)\,(-1)\cdot\rho(s,dx)\right)\bar\gamma(ds)\right]\\
&= \tfrac12\left[\int_S\left(\int_{R^+} I_{I_\varphi(C\cap(\partial U\times R^+))}(s,x)\rho(s,dx)\right)\bar\gamma(ds)\right.\\
&\qquad\left.+\int_S\left(\int_{R^+} I_{\bar I_\varphi(C)}(s,-x)\rho(s,dx)\right)\bar\gamma(ds)\right]\\
&= \tfrac12 F\left(I_\varphi\left(C\cap(\partial U\times R^+)\right)\right)\\
&\qquad+\tfrac12\int_{\partial U}\left(\int_{R^+} I_{\bar I_\varphi(C)}(\varphi(u),-x)\rho(\varphi(u),dx)\right)\bar\gamma_0(du)\\
&= \tfrac12 F\left(I_\varphi\left(C\cap(\partial U\times R^+)\right)\right)\\
&\qquad+\tfrac12\int_{\partial U}\left(\int_{R^+} I_{\bar I_\varphi(C)}\,(\varphi(-u),-x)\,\rho\,(\varphi(-u),dx)\right)\bar\gamma_0(du)\\
&= \tfrac12 F\left(I_\varphi\left(C\cap(\partial U\times R^+)\right)\right)\\
&\qquad+\tfrac12\int_{\partial U}\left(\int_{R^+} I_{\bar I_\varphi(-C)}\,(\varphi(u),x)\,\rho\,(\varphi(u),dx)\right)\bar\gamma_0(du)
\end{aligned}
$$

(using (4.12) and (4.13))

$$
= \tfrac12 F\left(I_\varphi\left(C\cap(\partial U\times R^+)\right)\right)+\tfrac12\left(I_\varphi\left(-C\cap(\partial U\times R^+)\right)\right).\ \blacksquare
$$

Now we are ready to state and prove our promised result providing the useful representation, similar to (4.1), of an arbitrary Lévy measure on ℓ_2.

THEOREM 4.2. *(a) Let M be a Lévy measure on ℓ_2; then F is the unique measure on $\mathcal{B}(S\times R^+)$ satisfying*

$$
M = F\circ\Psi^{-1}; \tag{4.17}
$$

(hence, from (4.7) and (4.17)) we have the desired representation of M : for every $D \epsilon \mathcal{B}(\ell_2 \setminus \{0\})$,

$$(4.18) \qquad M(D) = \int_S \left(\int_{R^+} I_D \left(x \varphi^{-1}(s) \right) \rho(s, dx) \right) \gamma(ds);$$

more generally,

$$(4.19) \qquad \int_{\ell_2 \setminus \{0\}} f dM = \int_S \left(\int_{R^+} f(x \varphi^{-1}(s)) \rho(s, dx) \right) \gamma(ds),$$

whenever either $f \geq 0$ or $\int_{\ell_2 \setminus \{0\}} |f| dM$ is finite, in the second case f can be complex.

(b) If M is symmetric, then $\overline{F}$ is the unique measure on $\mathcal{B}(S \times R_0)$ satisfying the symmetry condition (4.15) and

$$(4.20) \qquad M = \overline{F} \circ \overline{\Psi}^{-1};$$

and, in addition to (4.18), M also admits the representation:

$$(4.21) \qquad M(D) = \int_S \left(\int_{R_0} I_D \left(x \varphi^{-1}(s) \right) \overline{\rho}(s, dx) \right) \overline{\gamma}(ds),$$

for every $D \epsilon \mathcal{B}(\ell_2 \setminus \{0\})$; and the analog of (4.19) also holds.

PROOF OF (a): From (4.4), (4.6) and (4.7) we have, for any $C \epsilon \mathcal{B}(S \times R^+)$,

$$\begin{aligned} F(C) &= \int_{\partial U} \left(\int_{R^+} I_C \left(\varphi(u), x \right) \rho \left(\varphi(u), dx \right) \right) \gamma_0(du) \\ &= \int_{\partial U} \left(\int_{R^+} I_C(\varphi(u), x) \left[\min(1, x^2) \right]^{-1} q(u, dx) \right) \gamma_0(du) \\ &= \iint_{\partial U \times R^+} I_C(\varphi(u), x) \left[\min(1, x^2) \right]^{-1} d\Gamma_0 \\ &= \iint_{\partial U \times R^+} I_{I_\varphi^{-1}(C)}(u, x) \left[\min(1, x^2) \right]^{-1} d\Gamma_0 . \end{aligned}$$

On the other hand, for any $D \epsilon \mathcal{B}(\ell_2 \setminus \{0\})$,

$$\begin{aligned} M(D) &= \int_D \left[\min(1, \|z\|^2) \right]^{-1} dM_0 \\ &= \iint_{\partial U \times R^+} I_{\Psi_0^{-1}(D)}(u, x) \left[\min(1, |x|^2) \right]^{-1} d\Gamma_0 , \end{aligned}$$

by (4.3) and the change of variable formula. Hence, since $I_\varphi^{-1} \circ \Psi^{-1}(D) = (\Psi \circ I_\varphi)^{-1}(D) = \Psi_0^{-1}(D)$, we have $M(D) = F(\Psi^{-1}(D))$; proving (4.17). The proof of (4.18), as noted in the statement of the proposition, now follows from (4.7), the change of variable formula and the fact that $\Psi(s,x) = x\varphi^{-1}(s)$. The proof of (4.19) follows from (4.18) and the standard limiting arguments (see, e.g. [**1**, p. 104]). Finally, the proof of uniqueness of F is trivial, since the map Ψ is a Borel isomorphism between $S \times R^+$ and $\ell_2 \setminus \{0\}$.

PROOF OF (b): To prove (4.20), we use (4.16) and the facts $\overline{\Psi} = \overline{\Psi}_0 \circ \bar{I}_\varphi^{-1}$ and $\Psi = \Psi_0 \circ I_\varphi^{-1}$ (look at the Diagrams 4.1). Thus, for any $D \epsilon \mathcal{B}(\ell_2 \setminus \{0\})$,

$$\begin{aligned}
\overline{F}\left(\overline{\Psi}^{-1}(D)\right) &= \overline{F}\left(\bar{I}_\varphi\left(\overline{\Psi}_0^{-1}(D)\right)\right) \\
&= \tfrac{1}{2}\Big[F\left(I_\varphi\left(\overline{\Psi}_0^{-1}(D) \cap (\partial U \times R^+)\right)\right) \\
&\qquad + \overline{F}\left(I_\varphi\left(-\overline{\Psi}_0^{-1}(D) \cap (\partial U \times R^+)\right)\right)\Big] \\
&= \tfrac{1}{2}\left[F\left(I_\varphi\left(\Psi_0^{-1}(D)\right)\right) + F\left(I_\varphi\left(\Psi_0^{-1}(-D)\right)\right)\right] \\
&= \tfrac{1}{2}F\left(\Psi^{-1}(D)\right) + \tfrac{1}{2}F\left(\Psi^{-1}(-D)\right) \\
&= \tfrac{1}{2}\left[M(D) + M(-D)\right] = M(D),
\end{aligned}$$

(by part (a) and symmetry of M). To see the uniqueness of $\overline{F}$; suppose $\overline{F}_1$ is some other measure on $\mathcal{B}(S \times R_0)$ satisfying $\overline{F}_1(\bar{I}_\varphi(C)) = \overline{F}_1\left(\bar{I}_\varphi(-C)\right)$ and $M = \overline{F}_1 \circ \overline{\Psi}^{-1}$. Then, for any $D \epsilon \mathcal{B}(\ell_2 \setminus \{0\})$,

$$\begin{aligned}
M(D) &= \overline{F}_1\left(\overline{\Psi}^{-1}(D) \cap (S \times R^+)\right) + \overline{F}_1\left(\overline{\Psi}^{-1}(D) \cap (S \times R^-)\right) \\
&= \overline{F}_1\left(\Psi^{-1}(D)\right) + \overline{F}_1\left(\bar{I}_\varphi\left(\overline{\Psi}_0^{-1}(D) \cap (\partial U \times R^-)\right)\right) \\
&= \overline{F}_1\left(\Psi^{-1}(D)\right) + \overline{F}_1\left(\bar{I}_\varphi\left(\overline{\Psi}_0^{-1}(-D) \cap (\partial U \times R^+)\right)\right) \\
&= \overline{F}_1(\Psi^{-1}(D)) + \overline{F}_1\left(I_\varphi\left(\Psi_0^{-1}(-D)\right)\right) \\
&= \overline{F}_1\left(\Psi^{-1}(D)\right) + \overline{F}_1\left(\Psi^{-1}(D)\right) \\
&= 2\overline{F}_1\left(\Psi^{-1}(D)\right),
\end{aligned}$$

where we used the symmetry condition of $\overline{F}_1$ twice and the facts $\overline{\Psi} = \overline{\Psi}_0 \circ \bar{I}_\varphi^{-1}$ and $\Psi = \Psi_0 \circ I_\varphi^{-1}$. Thus $\overline{F}_1\left(\Psi^{-1}(D)\right) = \frac{1}{2}M(D)$; consequently, using uniqueness of F of part (a), we have $\overline{F}_1 = \frac{1}{2}F$ on $\mathcal{B}(S \times R^+)$. Using this, (4.16) and the symmetry condition of

$\overline{F}_1$, we have, for any $C \epsilon \mathcal{B}(S \times R_0)$,

$$\begin{aligned}\overline{F}_1\left(\bar{I}_\varphi(C)\right) &= \overline{F}_1\left(\bar{I}_\varphi(C) \cap (S \times R^+)\right) + \overline{F}_1\left(\bar{I}_\varphi(C) \cap (S \times R^-)\right) \\ &= \overline{F}_1\left(I_\varphi\left(C \cap (\partial U \times R^+)\right)\right) + \overline{F}_1\left(\bar{I}_\varphi\left(-C \cap (\partial U \times R^+)\right)\right) \\ &= \tfrac{1}{2} F\left(I_\varphi\left(C \cap (\partial U \times R^+)\right)\right) + \tfrac{1}{2} F\left(I_\varphi\left(-C \cap (\partial U \times R^+)\right)\right) \\ &= \overline{F}\left(\bar{I}_\varphi(C)\right);\end{aligned}$$

consequently $\overline{F}_1 = \overline{F}$ on $\mathcal{B}(S \times R_0)$. The proof of (4.21) is now a consequence of (4.10), change of variable formula and the fact that $\Psi(s, x) = x\varphi^{-1}(s)$; finally, the proof of the analog of (4.19) follows as noted in part (a). ∎

As we noted in the introductory remarks of this section, our representations of the Lévy measure M, obtained in the above theorem, is quite satisfactory with respect to the question: Do the measures $\rho(s, \cdot)$, $\bar{\rho}(s, \cdot)$, $F_A(\cdot)$ and $\overline{F}_A(\cdot)$ inherit properties of M? We address this question in Proposition 4.4 for three important classes of Lévy measures M; and show that M belongs to a fixed class of Lévy measures on ℓ_2 if and only if $\rho(s, \cdot)$ belongs to the same class of Lévy measures on R, for almost all s; similar result holds when $\rho(s, \cdot)$ is replaced by any of the other three measures. The methods of proof of this proposition suggest that one can possibly prove similar results for other classes of Lévy measures.

To facilitate the presentation of this result, we first introduce few more notations, and then prove a lemma which will be needed for the proof of the proposition. The contents of the lemma are essentially known but there is no single source to which reference can be made. For this reason and for completeness we include this lemma.

Let H denote a finite or infinite dimentional real separable Hilbert space. Then, we denote, by $\mathcal{M}_1(H)$, the set of all S(r, α) Lévy measures on H, by $\mathcal{M}_2(H)$, the set of all S(α) Lévy measures on H and, by $\mathcal{M}_3(H)$, the set of all SD Lévy measures on H. We recall that, for a given Lévy measure M on H, the following are well known:

(4.22) $\quad M \epsilon \mathcal{M}_1(H) \iff rM = r^{\frac{1}{\alpha}} \cdot M,$

(4.23) $\quad M \epsilon \mathcal{M}_2(H) \iff tM = t^{\frac{1}{\alpha}} \cdot M, \quad$ and for all $t \epsilon (0, 1]$,

(4.24) $\quad M \epsilon \mathcal{M}_3(H) \iff t \cdot M \leq M, \quad$ for all $t \epsilon (0, 1]$.

We also recall that if μ is and i.d. measure on H with Lévy measure M; the Lévy measures of μ^s, the s-th roots of μ, and $s \cdot \mu$, $s > 0$, are, respectively, sM and $s \cdot M$. We will use these facts in the proof of the lemma below. Now we are ready to state the lemma, which says that the interval $(0,1]$ in (4.23) and (4.24) can be replaced by any countable dense subset of it.

LEMMA 4.3. *Let M be a Lévy measure on H and T any countable dense subset of* $(0,1]$, *then $M \epsilon \mathcal{M}_2(H)$ (resp. $M \epsilon \mathcal{M}_3(H)$) $\Longleftrightarrow tM = t^{\frac{1}{\alpha}} \cdot M$ (resp. $t \cdot M \leq M$), for every $t \epsilon T$.*

PROOF: Let μ be the prob. measure on H with the ch. function

$$\hat{\mu}(y) = \exp \int_H \left(e^{i\langle z,y \rangle} - 1 - i \langle \tau(z), y \rangle \right) M(dz). \tag{4.25}$$

Now assume $tM = t^{\frac{1}{\alpha}} \cdot M$, for all $t \epsilon T$. This implies (use ch. functions)

$$\mu^t = t^{\frac{1}{\alpha}} \cdot \mu * \delta_{x(t)}, \qquad \text{for all } t \epsilon T, \tag{4.26}$$

where $x(t) \epsilon H$. Now let t_0 be an arbitrary element of $(0,1)$, choose a sequence $\{t_n\}$ from T such that $t_n \to t_0$; then $t_n^{\frac{1}{\alpha}} \cdot \mu \xrightarrow{w} t_0^{\frac{1}{\alpha}} \cdot \mu$ and, by the continuity of $\{\mu^s : s > 0\}$, $\mu^{t_n} \xrightarrow{w} \mu^{t_0}$. Hence, using (4.26), we also have $\delta_{x(t_n)} \xrightarrow{w} \delta_{x(t_0)}$. Showing $\mu^{t_0} = t_0^{\frac{1}{\alpha}} \cdot \mu \times \delta_{x(t_0)}$; thus we have $\mu^t = t^{\frac{1}{\alpha}} \cdot \mu * \delta_{x(t)}$, for all t. Hence $tM = t^{\frac{1}{\alpha}} \cdot M$, for all $t \epsilon (0,1]$, or $M \epsilon \mathcal{M}_2(H)$. The converse part of course is trivial.

The proof in the self decomposable case is similar: The condition $t \cdot M \leq M$, for $t \epsilon T$, implies $M = t \cdot M + M_t$, where $M_t = M - t \cdot M$ is also a Lévy measure (as $0 \leq M_t \leq M$). This, in turn implies: $\mu = t \cdot \mu * \nu_t$, for all $t \epsilon T$, where μ is as in (4.25) and ν_t an i.d. measure with Lévy measure M_t. Now, as in the above proof, it follows that $\mu = t \cdot \mu * \nu_t$, for all $t \epsilon (0,1]$. This implies $M \geq t \cdot M$, for all $t \epsilon (0,1]$. ∎

PROPOSITION 4.4. *Let M be a Lévy measure on ℓ_2; and $\rho(s,\cdot)$, $F_A(\cdot)$ and γ be the measures related to M as defined prior to Proposition 4.1. Then, for any fixed $i = 1,2,3$, $M \epsilon \mathcal{M}_i(H) \Longleftrightarrow$ off a γ-null set, $\rho(s,\cdot) \epsilon \mathcal{M}_i(R) \Longleftrightarrow F_A(\cdot) \epsilon \mathcal{M}_i(R)$, for all $A \epsilon \mathcal{B}(\partial U)$.*

PROOF: Throughout this proof, we denote by A, B and D the generic elements of $\mathcal{B}(\partial U)$, $\mathcal{B}(R^+)$ and $\mathcal{B}(\ell_2 \setminus \{0\})$, respectively. First we observe, from (4.18), that for any $a > 0$

$$\begin{aligned} a \cdot M(D) &= \int_S \left(\int_{R^+} I_{a^{-1}D} \left(x\varphi^{-1}(s) \right) \rho(s,dx) \right) \gamma(ds) \\ &= \int_S \left(\int_{R^+} I_D \left(ax\varphi^{-1}(s) \right) \rho(s,dx) \right) \gamma(ds) \\ &= \int_S \left(\int_{R^+} I_D \left(x\varphi^{-1}(s) \right) a \cdot \rho(s,dx) \right) \gamma(ds); \end{aligned} \tag{4.27}$$

and, if $D = \Psi(A \times B)$, then (recalling $\Psi^{-1}(z) = \left(\varphi\left(\frac{z}{\|z\|}\right), \|z\| \right)$ we get, from (4.27),

$$
(4.28) \qquad a \cdot M(D) = \int_S \left(\int_{R^+} I_{A\times B}(s,x)\, a \cdot \rho(s,dx) \right) \gamma(ds)
$$

$$
= \int_S \left(\int_{R^+} I_{A\times a^{-1}B}(s,x) \rho(s,dx) \right) \gamma(ds)
$$

$$
= \int_A \rho\left(s, a^{-1}B\right) \gamma(ds)
$$

$$
= a \cdot F_A(B).
$$

Further, for such a D,

$$
(4.29) \qquad aM(D) = \int_S \left(\int_{R^+} I_D\left(x\varphi^{-1}(s)\right) a\rho(s,dx) \right) \gamma(ds)
$$

$$
= \int_S \left(\int_{R^+} I_{A\times B}(s,x)\, a\rho(s,dx) \right) \gamma(ds)
$$

$$
= aF_A(B).
$$

Now we are ready to prove the proposition for $i = 1$. Let $M \epsilon \mathcal{M}_1(H)$; hence, by (4.22), $rM = r^{\frac{1}{\alpha}} \cdot M$. Therefore, by (4.28) and (4.29), $rF_A(\cdot) = r^{\frac{1}{\alpha}} \cdot F_A(\cdot)$, for all A; showing $F_A(\cdot)\epsilon\mathcal{M}_1(R)$. Now let $F_A(\cdot)\epsilon\mathcal{M}_1(R)$, for all A; then, from (4.28) and (4.29) again, $r\rho(s,B) = r^{\frac{1}{\alpha}} \cdot \rho(s,B)$ a.e. $[\gamma]$, for every fixed B. But, as $\mathcal{B}(R)$ is countably generated, $r\rho(s,dx) = r^{\frac{1}{\alpha}} \cdot \rho(s,dx)$, off a γ-null set. Showing $\rho(s,\cdot)\epsilon\mathcal{M}_1(R)$, off a γ-null set. Finally, if $\rho(s,\cdot)\epsilon\mathcal{M}_1(R)$, off a γ-null set, we have, from (4.27), that $rM = r^{\frac{1}{\alpha}} \cdot M$ or that $M\epsilon\mathcal{M}_1(\ell_2)$.

Now we consider the case $i = 2$. In view of the arguments used above, the only part that needs a justification is the proof of $F_A(\cdot)\epsilon\mathcal{M}_2(H)$ implies $\rho(u,\cdot)\epsilon\mathcal{M}_2(R)$, off a γ-null set. Let T be a countable dense subset of $(0,1]$. Then, assuming $F_A(\cdot)\epsilon\mathcal{M}_2(H)$ and using (4.28) and (4.29) and the fact that $\mathcal{B}(R)$ is countably generated, we have, for every $t\epsilon T$, $t\rho(s,\cdot) = t^{\frac{1}{\alpha}} \cdot \rho(s,\cdot)$ off a γ-null set N_t. Hence, same is true, for all $t\epsilon T$, off the γ-null set $N \equiv \bigcup_{t\epsilon T} N_t$. Therefore, by Lemma 4.3, $\rho(s,\cdot)\epsilon\mathcal{M}_2(R)$, for all $s \notin N$.

Finally, we consider the case $i = 3$; once again the only nontrivial part is to show $\rho(u,\cdot)\epsilon\mathcal{M}_3(R)$ off a γ-null set assuming $F_A(\cdot)\epsilon\mathcal{M}_3(R)$, for all A. To prove this let $F_A(\cdot)\epsilon$ $\mathcal{M}_3(R)$, for all A; and let $\mathcal{A} = \{(s,t),\ 0 < s < t < \infty, s,t \text{ rational}\}$ and T be a countable dense subset of (0,1). Then, for every fixed t and $B\epsilon\mathcal{A}$, we have

$$
t \cdot F_A(B) = \int_A t \cdot \rho(s,B)\gamma(ds) \geq \int_A \rho(s,B)\gamma(ds) = F_A(B),
$$

for all A. Hence, since $\rho(\cdot,B)$ and $t \cdot \rho(\cdot,B)$ are finite and T and $\mathcal{A}$ are countable, there exists a γ-null set N_0 such that if $s \notin N_0$ then $t \cdot \rho(s,B) \leq \rho(s,B)$, for all $B\epsilon\mathcal{A}$ and all $t\epsilon T$. This, along with Lemma 4.3, show $\rho(s,\cdot)\epsilon\mathcal{M}_3(R)$ (note $\rho(s,R_0) = 0$), off the set N_0. ∎

REMARK 4.5. *If M is symmetric, then exactly the same proofs as above show: For every fixed $i = 1,2,3$, $M \epsilon \mathcal{M}_i(H) \Leftrightarrow \overline{p}(s,\cdot)\epsilon\mathcal{M}_i(R)$, off a $\overline{\gamma}$-null set $\Leftrightarrow \overline{F}_A(\cdot)\epsilon\mathcal{M}_i(R)$, for all A.*

Now we are ready to obtain the *main* results of this section; namely, the spectral representations of various i.d. discrete processes. We begin with a few more necessary notations and definitions:

Let μ be an **ID** prob. measure on ℓ_2 with the Lévy representation: $\mu \sim [z_0,\ K,\ M]$, where $z_0 \epsilon \ell_2$, K is the covariance operator and M is the Lévy measure of μ (which will always be assumed symmetric if μ is symmetric). Let $K(y) = \sum_j \beta_j \langle e_j, y\rangle e_j$, where $\beta_j \geq 0$, $\sum_j \beta_j < \infty$ and $\{e_j\}$ is an orthonormal set in ℓ_2. Define the finite measures $\tilde{\nu}_0$, $\tilde{\nu}_1$ on ∂U and ν_0, ν_1 on S by:

$$\tilde{\nu}_0 = \left\{ \begin{array}{lll} \|z_0\| \delta_{\left\{\frac{z_0}{\|z\|}\right\}}, & \text{if} & z_0 \neq \theta \\ 0, & \text{if} & z_0 = \theta \end{array} \right\}, \qquad \tilde{\nu}_1 = \sum_j \beta_j \delta_{\{e_j\}},$$

and

$$\nu_0 = \tilde{\nu}_0 \circ \varphi^{-1}, \qquad \nu_1 = \tilde{\nu}_1 \circ \varphi^{-1}.$$

Let Λ be the **ID** r. measure on $(S, \mathcal{B}(S))$ with parameters $(\nu_0,\ \nu_1,\ F_\cdot)$ (see Proposition 2.1), we will refer to Λ as the associated r. measure of μ or of $[z_0,\ K,\ M]$. If μ is symmetric (hence M is also symmetric according to our convention), then the r. measure $\overline{\Lambda}$ with parameters $(0,\ \nu_1,\ \overline{F}_\cdot)$ will also be referred to as the associated r. measure of μ or of $[0,\ K,\ M]$. (Here of course, $F_\cdot$ and $\overline{F}_\cdot$ are the Lévy measures on R as in Lemma 4.1). In order to keep similar notations, we will write ν for the measure γ defined in (4.6); and, in the symmetric case, we will use the notation $\overline{\nu}_1$ for ν_1 and $\overline{\nu}$ for ν. Note that the control measures of Λ and $\overline{\Lambda}$ are, respectively, given by $\lambda = \nu_0 + \nu_1 + \nu$ and $\overline{\lambda} = \overline{\nu}_1 + \overline{\nu}$ (see (2.3)).

The above notations and conventions will remain fixed throughout this and the next section; in addition, we will denote, by π_n, the n-th coordinate projection on ℓ_2 and, by g_n, the Borel map on S defined by $g_n(s) = \pi_n(\varphi^{-1}(s))$. In the following lemma, we record three integral identities, these will be needed in the sequal. The proofs of the first two are straightforward and the proof of the last is a direct consequence of (4.19).

LEMMA 4.6. *Let $a_1, a_2, \ldots, a_n$ be n-real numbers, then*

$$\int_S \left(\sum_{j=1}^{n} a_j g_j(s) \right) \nu_0(ds) = \sum_{j=1}^{n} a_j \pi_j(z_0), \tag{4.30}$$

$$\int_S \left(\sum_{j=1}^{n} a_j g_j(s) \right)^2 \nu_1(ds) = \sum_k \beta_k \left(\sum_{j=1}^{n} a_j \pi_j(e_k) \right)^2 = \langle Ky, y \rangle, \tag{4.31}$$

$$\int_S \left(\int_{R^+} \min\left(1, g_n^2(s) x^2\right) \rho(s, dx) \right) \nu(ds) = \int_{\ell_2} \min\left(1, \pi_n^2(z)\right) M(dz). \tag{4.32}$$

We are now ready to state and prove the most basic result of this section; this, as we will see, essentially provides the spectral representations of all discrete **ID** processes.

THEOREM 4.7. *Let μ be an **ID** prob. measure on ℓ_2 with Lévy representation $[z_0, K, M]$ and let Λ be the associated **ID** r. measure on S. Let $Y_n(\cdot) = \pi_n(\cdot)$ $n = 1, 2, \ldots$, be the r. variables on $(\ell_2, \mathcal{B}(\ell_2), \mu)$. Then the functions g_n's are Λ-integrable, and we have*

$$\{Y_n\} \stackrel{d}{=} \left\{ \int_S g_n d\Lambda \right\}.$$

PROOF: In order to see that g_n's are Λ-integrable, we have to verify (i)–(iii) of Theorem 2.3. But, in view of (4.31) and (4.32), and the fact that $\int_{\ell_2} \min(1, \pi_n(z)^2) M(dz) \leq \int_H \min\left(1, \|z\|^2\right) M(dz) < \infty$, we need only verify (i). Thus, in view of (4.30), we need only to verify that

$$\int_S \left(\int_{R^+} \left[\tau\left(g_n(s) x\right) - g_n(s) \tau(x) \right] \rho(s, dx) \right) \nu(ds)$$

is finite. But this follows since the absolute value of the integrand is no more than $(1 + |g_n(s)|) \max\left(1, g_n^2(s)\right)$ and since $|g_n(s)| \leq 1$ and ν is finite.

In order to show $\{Y_n\} \stackrel{d}{=} \{\int_S g_n d\Lambda\}$, we must prove

$$\hat{\mathcal{L}} \left(\sum_{j=1}^{k} a_j Y_j \right)(1) = \hat{\mathcal{L}} \left(\sum_{j=1}^{k} a_j \int_S g_j d\Lambda \right)(1), \tag{4.33}$$

for every fixed k, and $a_1, \ldots, a_k$ real.

Now, the left side of (4.33)

$$= E \exp\left(i \sum_{j=1}^{k} a_j Y_j \right)$$

$$= \int_{\ell_2} \exp\left(i \sum_{j=1}^{k} a_j \pi_j(z) \right) d\mu = \int_{\ell_2} e^{i\langle z,y\rangle} d\mu$$

$$= \exp\left\{ i\langle z_0, y\rangle - \tfrac{1}{2}\langle Ky, y\rangle + \int_{\ell_2} \left(e^{i\langle z,y\rangle} - 1 - i\langle \tau(z), y\rangle \right) dM \right\},$$

where $y = (a_1, \ldots, a_k,\ 0, 0, \ldots)$; and the right side of (4.33), by (2.5),

$$= \exp\left\{ i \int_S \left(\sum_{j=1}^{k} a_j g_j(s) \right) \nu_0(ds) - \tfrac{1}{2} \int_S \left(\sum_{j=1}^{k} a_j g_j(s) \right)^2 \nu_1(ds) \right.$$
$$\left. + \int_S \left[\int_{R^+} \left(e^{ix\left(\sum_{j=l}^{k} a_j g_j(s)\right)} - 1 - i\left(\sum_{j=1}^{k} a_j g_j(s) \tau(x) \right) \right) \rho(s, dx) \right] \nu(ds) \right\}.$$

Thus, recalling (4.30) and (4.31), we need only to verify that

$$\int_{\ell_2} \left(e^{i\langle z,y\rangle} - 1 - i\langle \tau(z), y\rangle \right) dM \tag{4.34}$$
$$= \int_S \left[\int_{R^+} \left(e^{ix\left(\sum_{j=1}^{k} a_j g_j(s)\right)} - 1 - i\left(\sum_{j=1}^{k} a_j g_j(s) \right) \tau(x) \right) \rho(s, dx) \right] \nu(ds).$$

But, from (4.19), the left side of this equation

$$= \int_S \left[\int_{R^+} \left(e^{i\langle x\varphi^{-1}(s), y\rangle} - 1 - i\left\langle \tau\left(x\varphi^{-1}(s)\right), y\right\rangle \right) \rho(s, dx) \right] \nu(ds)$$

$$= \int_S \left[\int_{R^+} \left(e^{ix\sum_{j=1}^{k} a_j \pi_j(\varphi^{-1}(s))} - 1 - i\left(\sum_{j=1}^{k} a_j \pi_j\left(\varphi^{-1}(s)\right) \right) \tau(x) \right) \rho(s, dx) \right] \nu(ds),$$

since

$$\tau\left(x\varphi^{-1}(s)\right) = \left\{ \begin{array}{lll} x\varphi^{-1}(s), & \text{if} & 0 < \|x\varphi^{-1}(s)\| = x \le 1 \\ \frac{x\varphi^{-1}(s)}{\|x\varphi^{-1}(s)\|} = \varphi^{-1}(s), & \text{if} & x > 1 \end{array} \right\}.$$

Thus, since $\pi_j\left(\varphi^{-1}(s)\right) = g_j(s)$, we have the validity of (4.34). ∎

Now we show, in the following corollary, that the above theorem yields spectral representations of all discrete **ID** process without having to center or symmetrize the processes.

COROLLARY 4.8. *Let $\{X_n\}$ be an* **ID** *process satisfying $E|X_n|^q < \infty$, for some $q \geq 0$. Let $b_n > 0$ be such that $Y = \{b_n X_n\} \epsilon \ell_2$ almost surely. Let μ be the law of Y on ℓ_2 (which is* **ID** [20]*); and let Λ be the associated* **ID** *r. measure on S, then $f_n = b_n^{-1} g_n$'s belong to $L_{\Phi_p}(S;\lambda)$, for any $0 \leq p \leq q$; and*

$$\{X_n\} \stackrel{d}{=} \left\{ \int_S f_n d\Lambda \right\}. \tag{4.35}$$

PROOF: Clearly f_n's are Λ-integrable as g_n's are. To see that (4.35) holds; let $a_1, \ldots, a_k$ be real numbers, then, noting that $\mathcal{L}(Y_1, \ldots, Y_k) = \mathcal{L}(\pi_1(\cdot), \ldots, \pi_k(\cdot))$ and using the above theorem, we have

$$\begin{aligned}
\hat{\mathcal{L}}\left(\sum_{j=1}^{k} a_j X_j\right)(1) &= \hat{\mathcal{L}}\left(\sum_{j=1}^{k} \frac{a_j}{b_j} Y_j\right)(1) \\
&= \hat{\mathcal{L}}\left(\sum_{j=1}^{k} \frac{a_j}{b_j} \int_S g_j d\Lambda\right)(1) \\
&= \hat{\mathcal{L}}\left(\sum_{j=1}^{k} a_j \int_S f_j d\Lambda\right)(1).
\end{aligned}$$

showing the validity of (4.35). Finally, since $E|X_n|^p = E|\int_S f_n d\Lambda|^p < \infty$, for any $0 \leq p \leq q$, we have from Theorem 3.3 that $f_n \epsilon L_{\Phi_p}(S;\lambda)$. ∎

Before we can assert that the above theorem yields known spectral representations for discrete stable and semistable processes, we need one more result:

LEMMA 4.9. *Let $Y \equiv \{Y_n\}$ be an* **ID** *process with almost all sample paths in ℓ_2. Then $\mu \equiv \mathcal{L}(Y)$ is an $\mathrm{S}(\alpha)$ (resp. $\mathrm{S}(r,\alpha)$;* **SD***) prob. measure if Y is an $\mathrm{S}(\alpha)$ (resp. $\mathrm{S}(r,\alpha)$;* **SD***) process. Further, if Y is centered $\mathrm{S}(\alpha)$ (resp. $\mathrm{S}(r,\alpha)$) process then μ is a centered $\mathrm{S}(\alpha)$ (resp. $\mathrm{S}(r,\alpha)$) prob. measure.*

PROOF: A proof of the last part in the centered $\mathrm{S}(r,\alpha)$ case is provided in [20]. Similar proof works in the other cases. We outline the proof in the **SD** case. Denote by $\pi_{1,\ldots,n}$ the natural projection from ℓ_2 onto R^n; and let $0 < a < 1$ be fixed. First observe $\mu \circ \pi_{1,\ldots,n}^{-1} = \mathcal{L}(Y_1, \ldots, Y_n)$ and $(a \cdot \mu) \circ \pi_{1,\ldots,n}^{-1} = a \cdot (\mu \circ \pi_{1,\ldots,n}^{-1}) = a \cdot \mathcal{L}(Y_1, \ldots, Y_n)$. Hence, as Y is a **SD** process, there exists a unique prob. measure ν_n on R^n (recall (1.3)) satisfying

$$\mu \circ \pi_{1,\ldots,n}^{-1} = (a \cdot \mu) \circ \pi_{1,\ldots,n}^{-1} * \nu_n. \tag{4.36}$$

Now, using Kolmogorov's extension theorem, we construct a unique prob. measure ν on R^∞ with $\nu \circ \pi_{1,\dots,n}^{-1} = \nu_n$. Using (4.36) and viewing the measures μ and $a \cdot \mu$ on R^∞ and using ch. functions, we find

$$\mu = a \cdot \mu * \nu$$

on R^∞. But, then $1 = \mu(\ell_2) = \int_{\ell_2} \nu(\ell_2 + x) a \cdot \mu(dx)$; hence $\nu(\ell_2) = 1$. ∎

In view of Corollary 4.8, Lemma 4.9 and Proposition 4.4, we obtain all known spectral representation for *discrete* parameter stable and semistable processes [**2**, **7**, **13**, **20**, **27**, **28**] without having to center or to symmetrize the process. Of course, these three results put together also yield similar spectral representations for **SD** processes.

V. Spectral Representations of Centered Continuous Parameter ID Processes

Unlike the discrete case, our methods, unfortunately, do not allow us to obtain spectral representations for arbitrary continuous parameter **ID** processes. However, if the process satisfies some additional conditions then one can indeed obtain spectral representations for such a process. These, besides providing spectral representations for new classes of **ID** processes, also yield, in a unified way, all previously known spectral representations for stable and semistable processes. We address these points in this section; as we noted in the previous section, the notations and convention of the previous section are in effect in this section as well.

Let T be an arbitrary set and $X = \{X_t : t \epsilon T\}$ be an **ID** process which is separable in $L_q(\equiv L_q(\Omega; P))$, $0 \le q < \infty$; (i.e., there exists a countable set $T_0 = \{t_n\}$ of T such that, for every $t \epsilon T$, there is sequence $\{s_m\} \subseteq T_0$ with $X_{s_m} \longrightarrow X_t$ in L_q). Recall that if T is a separable metric space and X is L_q-continuous than X is separable in L_q.

If $q = 0$, we choose $b_n > 0$ so that

$$\sum_{n=1}^{\infty} |b_n X_n|^2 < \infty,$$

almost surely, where $X_n = X_{t_n}$; if $q > 0$ (hence $E|X_t|^q < \infty$, for all t), then we choose $b_n > 0$ satisfying, additionally,

$$E\left(\sum_{n=1}^{\infty} |b_n X_n|^2\right)^{\frac{q}{2}} < \infty. \tag{5.1}$$

A choice of b_n's satisfying (5.1) is possible follows, since

$$\left(\sum_{n=1}^{\infty} |b_n X_n|^2\right)^{\frac{q}{2}} \leq \sum_{n=1}^{\infty} \left\{|b_n X_n|^2\right\}^{\frac{q}{2}} = \sum b_n^q |X_n|^q,$$

if $0 < q \leq 2$, and

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |b_n X_n|^2\right)^{\frac{q}{2}} &= \left(\sum_{n=1}^{\infty} 2^{-n} \left|b_n 2^{\frac{n}{2}} X_n\right|^2\right)^{\frac{q}{2}} \\ &\leq \sum_{n=1}^{\infty} 2^{-n} b_n^q 2^{q\frac{n}{2}} |X_n|^q \\ &= \sum_{n=1}^{\infty} 2^{\left(\frac{q}{2}-1\right)n} b_n^q |X_n|^q,\end{aligned}$$

if $2 < q < \infty$. Thus, if $q > 0$, then we have that $\int_{\ell_2} \|z\|^q d\mu < \infty$; hence $\int_{\{\|z\|>1\}} \|z\|^q dM < \infty$, where M is the Lévy measure of μ, the law of $Y \equiv \{Y_n = b_n X_n\}$ on ℓ_2. Further, recalling (4.19), we have

$$\begin{aligned}\infty > \int_{\{\|z\|>1\}} \|z\|^q dM &= \int_S \left(\int_{R^+} \|x\varphi^{-1}(s)\|^q I\left(|x| > 1\right) \rho(s, dx)\right) \nu(ds) \\ &= \int_S \int_{R^+} |x|^q I\left(|x| > 1\right) \rho(s, dx) \nu(ds) \\ &\geq \int_{R^+} |x|^q I\left(|x| > 1\right) F_A(dx), \text{ for every } A. \end{aligned} \tag{5.2}$$

In the following, the above assumptions and notations will be in effect.

We shall obtain spectral representations for **ID** processes X which satisfy any one of the following additional assumptions:

(A-1) X is symmetric and $0 \leq q$, arbitrary,

(A-2) X is arbitrary (as above) and $1 \leq q$ with $E(X_t) = 0$,

(A-3) X is centered $\mathrm{S}(\alpha)$ or centered $\mathrm{S}(r, \alpha)$ $0 < \alpha < 2$ and $0 \leq q < \alpha$.

In order to obtain these spectral representations, we need to define additional r. measures (besides Λ in the previous section) associated to the p. measure μ (the law of Y). If X and q satisfy (A-1), we associate to μ the symmetric r. measure $\overline{\Lambda}_q$ with parameters $(0, \overline{\nu}_1, \overline{F}_.)$ (see the discussion prior to Lemma 4.6). If X and q satisfy (A-2), we associate to μ the r. measure Λ_q with parameters $(\nu_0,\ \nu_1,\ F_.)$, where ν_0 and ν_1 are given in (3.9) (recall that, in view of (5.2), $E\left|\Lambda_q(A)\right|^q < \infty$). If X is a centered $\mathrm{S}(\alpha)$ (resp. $\mathrm{S}(r, \alpha)$) process, then by

Lemma 4.9, μ is a centered $S(\alpha)$ (resp. $S(r,\alpha)$) p. measure; and, hence by Proposition 4.4, F is a $S(\alpha)$ (resp. $S(r,\alpha)$) Lévy measure. In this case, we associate to μ the r. measure Λ with parameters $(\nu_0,\ 0,\ F)$, where ν_0 is given in (3.9), if $1 < \alpha < 2$, and in (3.11), if $0 < \alpha < 1$. Note that, as follows from (3.9) and (3.11), Λ is a centered $S(\alpha)$ (resp. $S(r,\alpha)$) r. measure. Similar remark applies when X satisfies (A-1) and X is a centered $S(\alpha)$, $S(r,\alpha)$ or **SD** process; and when X satisfies (A-2) and X is a **SD** process.

Now we are ready to state and prove our main result of this section.

THEOREM 5.1. *Let $X = \{X_t : t\epsilon T\}$ be an* **ID** *L_q-separable process satisfying any one of (A-1)-(A-3) assumptions and let Λ be the corresponding* **ID** *r. measure with control measure λ. Then,*

$$X \stackrel{d}{=} \{\int_S f_t d\Lambda : t\epsilon T\}, \tag{5.3}$$

where, for every t, $f_t \epsilon L_{\Phi_q}(S;\lambda)$. Further, under assumption (A-3) Λ is a centered $S(\alpha)$ (resp. $S(r,\alpha)$) r. measure, if X is a centered $S(\alpha)$ (resp. $S(r,\alpha)$) r. process.

PROOF: The proofs under any one of the three assumptions are similar and use Propositions 3.6 and 3.8, the methods of proof of Theorem 4.7, and the L_q-separability of X. To exhibit the ideas of the proof, we outline the proof only under the assumption (A-2).

Since ν_0 is a finite measure, the proof of the fact that $g_n(s) = \pi_n(\varphi^{-1}(s))$, $n = 1,2,\ldots$, are Λ_q-integrable is exactly the same as in Theorem 4.7. Now we show that

$$\{\pi_n(\cdot)\} \stackrel{d}{=} \{\int_S g_n d\Lambda_q\}. \tag{5.4}$$

Fix $a_1,\ldots,a_k$, k real numbers; then recalling that

$$\int_{\ell_2} e^{i\langle z,y\rangle} d\mu = \exp\left[-\tfrac{1}{2}\langle Ky,y\rangle + \int_{\ell_2}\left\{e^{i\langle z,y\rangle} - 1 - i\langle z,y\rangle\right\} dM\right],$$

for every $y\epsilon\ell_2$, and, using (4.31) and (4.19), we have

$$\begin{aligned}\hat{\mathcal{L}}\left(\sum_{j=1}^{k} a_j\pi_j(\cdot)\right)(1) &= \exp\left[-\tfrac{1}{2}\int_S g^2(s)\nu_1(ds)\right.\\ &\qquad \left.+ \int_{\ell_2}\left\{e^{i\sum_{j=1}^{k} a_j\pi_j(z)} - 1 - i\sum_{j=1}^{k} a_j\pi_j(z)\right\} dM\right]\\ &= \exp\left[-\tfrac{1}{2}\int_S g^2(s)\nu_1(ds)\right.\\ &\qquad \left.+ \int_S\left\{\int_{\mathbf{R}^+}\left(e^{ixg(s)} - 1 - ixg(s)\right)\rho(s,dx)\right\}\nu(ds)\right],\end{aligned}$$

where $g(s) = \sum_{j=1}^k a_j g_j(s)$. On the other hand, by (2.5),

$$(5.5)\quad \hat{\mathcal{L}}\left(\int_S g(s)d\Lambda_q\right)(1) = \exp\left[\int_S g(s)\nu_0(ds) - \tfrac{1}{2}\int_S g^2(s)\nu_1(ds) + \int_S\left\{\int_{R^+}\left(e^{ixg(s)} - 1 - ig(s)\tau(x)\right)\rho(s,dx)\right\}\nu(ds)\right].$$

The first and last integral on the right side of (5.5) can be combined to see that $\hat{\mathcal{L}}(\int_S g(s)\, d\Lambda_q)$ (1) is equal to $\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j\pi_j(\cdot)\right)$ (1); proving (5.4). Now using the same argument as in Corollary 4.8, we observe that

$$\{X_{t_n} : t_n \epsilon T_0\} \stackrel{d}{=} \left\{\int_S f_{t_n} d\Lambda_q : t_n \epsilon T_0\right\},$$

where $f_{t_n} = b_n^{-1} g_n$. According to Proposition 3.6, we have that the map

$$L_{\Phi_q}(S,\lambda_q) \ni f \longmapsto \int_S f d\Lambda_q \epsilon L_q(\Omega, P)$$

is an isomorphism. Let $t\epsilon T$; choose a sequence $\{s_m\} \subseteq T_0$ such that $X_{s_m} \longrightarrow X_t$ in L_q. It follows that $\{\int_S f_{s_m} d\Lambda_q\}$ converges in L_q; hence, from Proposition 3.7, we have that there exists an f_t in $L_{\Phi_q}(S,\lambda_q)$ and that $\int_S f_{s_m} d\Lambda_q \longrightarrow \int_S f_t d\Lambda_q$ in L_q. Now, in order to prove (5.3), we must show

$$\mathcal{L}(X_{\ell_1},\ldots,X_{\ell_k}) = \mathcal{L}(\int_S f_{\ell_1} d\Lambda_q,\ldots,\int_S f_{\ell_k} d\Lambda_q)$$

for any fixed $\ell_1,\ldots,\ell_k \epsilon T$. But this follows from the usual limiting arguments: Fix real numbers $a_1,\ldots,a_k$ and choose $\{s_{ij}\} \subseteq T_0$ with $X_{s_{ij}} \to X_{\ell_i}$, $i = 1,\ldots,k$. Then, from what we have proved above, $\int_S f_{s_{ij}} d\Lambda_q \longrightarrow \int_S f_{\ell_i} d\Lambda_q$ in L_q, as $j \to \infty$, for each $i = 1,\ldots,k$; therefore $\sum_{i=1}^k a_i \int_S f_{s_{ij}} d\Lambda_q \longrightarrow \sum_{i=1}^k a_i \int_S f_{\ell_i} d\Lambda_q$ in L_q. Thus, since $\sum_{i=1}^k a_i X_{s_{ij}} \longrightarrow \sum_{i=1}^k a_i X_{\ell_i}$ in L_q and $\sum_{i=1}^k a_i \int_S f_{s_{ij}} d\Lambda_q \stackrel{d}{=} \sum_{j=1}^k a_i X_{s_{ij}}$, we have $\sum_{i=1}^k a_i \int_S f_{\ell_i} d\Lambda_q \stackrel{d}{=} \sum_{i=1}^k a_i X_{\ell_i}$. This completes the proof as $a_1,\ldots,a_k$ were arbitrary. ∎

Remark 5.2. (a) As noted in the introductory remarks, the above theorem obviously yields the known spectral representations for stable and semistable representations [**2**, **7**, **13**, **20**, **27**, **28**].

(b) One important point regarding the above theorem which is not explicitly stated but should be emphasized is the fact that the map

$$L_q(\Omega;P) \ni \sum_{j=1}^k a_j X_{t_j} \longmapsto \sum_{j=1}^k a_j f_{t_j} \epsilon L_{\Phi_q}(S;\lambda)$$

extends to topological isomorphism from the L_q-closure of the span of $\{X_t : t \epsilon T\}$ onto the closure of the span of $\{f_t : t \epsilon T\}$ in the space $L_{\Phi_q}(S;\lambda)$. This fact is important in that it would hopefully allow one, just as in the case of stable and Gaussian processes, to study the prediction problem and the structural properties of the process by making use of the above isomorphism and the rich structure of the (function) Musielak-Orlicz spaces $L_{\Phi_q}(S;\lambda)$.

(c) Theorem 5.1 raises the obvious question: For what other (besides those satisfying (A-1)-(A-3)) L_q-separable **ID** processes $X = \{X_t : t \epsilon T\}$, one can obtain spectral representations? A careful look at the proof of Theorem 5.1 reveals that one can obtain a spectral representation for any L_q-separable **ID** processes X for which the r. measure Λ associated to μ (the law of $Y \equiv \{b_n X_{t_n}\}$) can be chosen so that:

(i) $\{\pi_n(\cdot)\} \stackrel{d}{=} \{\int_S g_n d\Lambda\}$

(ii) the map $L_{\Phi_q}(S;\lambda) \ni f \longmapsto \int_S f d\Lambda \epsilon L_q(\Omega; P)$

is an isomorphism, where b_n, π_n, g_n are as in the theorem. Unfortunately, this criterion is not very satisfactory as the conditions (i) and (ii) are not explicitly given in terms of the given **ID** process X. Nevertheless, as we exhibited in the proof of Theorem 5.1, if more information is available about X this criterion can be successfully applied to obtain the spectral representation.

(d) Finally, if one is interested in obtaining spectral representations of arbitrary (i.e., those which are not L_q-separable) **ID** processes $X = \{X_t : t \epsilon T\}$, the methods used in Theorem 5.1 are not adequate. It appears that to handle such a problem one must replace the space ℓ_2 by a much larger linear space E, like $\mathbf{R}^T$, obtain a factorization of M, the Lévy measure of the law of X on E, similar to (4.18) and then, using the methods of Theorem 4.7, obtain a representation of X. At present, however, we are unable to see our way through clearly on this point; and we hope to shed more light on this in the future.

VI. Refinement of Spectral Representations in Distribution to Spectral Representations Which Hold Almost Surely

In this section, we shall show that the spectral representations of stochastic processes obtained in the previous sections can be modified so that the new representations hold almost surely. This, however, requires that the processes be redefined on a slightly larger prob. space. The possibility of such a refinement, by making use of the randomization lemma (Lemma 1.1 [**12**]), was suggested to us by O. Kallenberg. It is a great pleasure for both of us to thank Prof. Kallenberg for this suggestion. For our purposes, we shall need a slight generalization of the randomization lemma, which can be proven essentially by the same argument as Lemma 1.1 [**12**]. We omit this proof.

LEMMA 6.1. *Let ξ and η' be random elements defined on the prob. spaces (Ω, P) and (Ω', P'), and taking values in the spaces S and T, respectively, where S is a separable metric space while T is a Polish space. Assume that $\xi \stackrel{d}{=} f(\eta')$ for some Borel measurable function $f : T \to S$. Then there exists a random element $\eta \stackrel{d}{=} \eta'$ on the ("randomized") prob. space $(\Omega \times [0,1],\ P \times Leb)$ such that $\xi = f(\eta)$ a.s. $P \times Leb$.*

THEOREM 6.2. *Let $\{X_t : t \epsilon T\}$ be an **ID** stochastic process defined on a prob. space (Ω, P). Assume that*

$$\{X_t : t\epsilon T\} \stackrel{d}{=} \{\int_S f_t d\Lambda' : t\epsilon T\},$$

*where Λ' is an **ID** r. measure defined on a prob. space (Ω', P') and S is a Borel subset of a Polish space. Then there exists an **ID** r. measure Λ defined on the prob. space $(\Omega \times [0,1],$ $P \times Leb)$ such that*

$$\{\Lambda(A) : A \epsilon \mathcal{S}\} \stackrel{d}{=} \{\Lambda'(A) : A \epsilon \mathcal{S}\}$$

(here $\mathcal{S}$ is the Borel σ-algebra of S) and

$$X_t = \int_S f_t d\Lambda \qquad \text{a.s. } P \times Leb,$$

for every $t \epsilon T$.

PROOF: We have that $f_t \epsilon L_{\Phi_0}(S; \lambda)$ for every $t \epsilon T$, where λ is the control measure of Λ'. Since $\mathcal{S}$ is countably generated, L_{Φ_0} is separable. Hence there exists a set $T_0 = \{t_n\}_{n=1}^{\infty} \subset T$ such that $\{f_{t_n}\}_{n=1}^{\infty}$ is dense in $\{f_t\}_{t\epsilon T} \subset L_{\Phi_0}$. Define $\xi : \Omega \longrightarrow \mathbf{R}^\infty$ by

$$\xi(\omega) = (X_{t_1}(\omega),\ X_{t_2}(\omega),\ \ldots).$$

Choose $S_0 = \{A_j\}_{j=1}^{\infty}$ to be a countable algebra of sets such that $S_0 \subset S$ and $\sigma(S_0) = S$. Define $\eta' : \Omega \longmapsto \mathbf{R}^{\infty}$ by

$$\eta'(\omega') = (\Lambda'(A_1)(\omega'),\ \Lambda'(A_2)(\omega'),\ \ldots).$$

Since, for every $f \in L_{\Phi_0}$, there exists a sequence $\{g_k\}$ of simple S_0-measurable functions such that $g_k \longrightarrow f$ in L_{Φ_0}, we get, by Theorem 3.3, that $\int_S g_k \, d\Lambda' \to \int_S f \, d\Lambda'$ in prob. as $k \to \infty$. In particular, $\int_S f \, d\Lambda'$ is equal a.s. $[P']$ to some $\sigma\{\Lambda'(A_j) : j \geq 1\} = \sigma(\eta')$-measurable r. variable. Consequently, for every n, there exists a Borel function $\varphi_n : \mathbf{R}^{\infty} \longrightarrow \mathbf{R}$ such that

$$\int_S f_{t_n} \, d\Lambda' = \varphi_n(\eta') \qquad \text{a.s. } [P']. \tag{6.1}$$

Then, by the assumption of our theorem,

$$\{X_{t_n} : n \geq 1\} \stackrel{d}{=} \{\varphi_n(\eta') : n \geq 1\}$$

or

$$\xi \stackrel{d}{=} \Phi(\eta'),$$

where $\Phi : \mathbf{R}^{\infty} \longrightarrow \mathbf{R}^{\infty}$ is the Borel function defined by $\Phi(x) = (\varphi_1(x), \varphi_2(x), \ldots)$, $x \in \mathbf{R}^{\infty}$. In view of Lemma 6.1, there exists an $\mathbf{R}^{\infty}$-valued r. element η defined on $(\Omega \times [0,1],\ P \times Leb)$ such that $\eta \stackrel{d}{=} \eta'$ and $\xi = \Phi(\eta)$ a.s. $P \times Leb$. Put

$$\Lambda(A_j) = \eta_j, \qquad A_j \in S_0.$$

Since η' is the restriction of the r. measure Λ' to the algebra S_0 and $\eta \stackrel{d}{=} \eta'$, there exists a unique (modulo $P \times Leb$) extension of Λ to a r. measure on $\sigma(S_0) = S$ such that

$$\{\Lambda(A) : A \in S\} \stackrel{d}{=} \{\Lambda'(A) : A \in S\}. \tag{6.2}$$

By (6.1) we get

$$\varphi_n(\eta) = \int_S f_{t_n} \, d\Lambda \qquad \text{a.s. } P \times Leb;$$

which yields

$$X_{t_n} = \int_S f_{t_n} \, d\Lambda \qquad \text{a.s. } P \times Leb, \tag{6.3}$$

for every $n \geq 1$.

Let now $t \epsilon T$ be arbitrary. We can choose a sequence $\{t_{n(k)}\}_{k=1}^{\infty} \subset T_0$ such that $f_{t_{n(k)}} \longrightarrow f_t$ in L_{Φ_0}. By (6.2) and the assumption of our theorem,

$$(X_{t_{n(k)}},\ X_t) \stackrel{d}{=} (\int_S f_{t_{n(k)}} d\Lambda,\ \int_S f_t d\Lambda).$$

Since $\int_S f_{t_{n(k)}} d\Lambda \longrightarrow \int_S f_t d\Lambda$ in $P \times Leb$ as $k \longrightarrow \infty$, we get that $X_{t_{n(k)}} \rightarrow X_t$ in $P \times Leb$ as $k \rightarrow \infty$. By (6.3), $X_t = \int_S f_t d\Lambda$ a.s. $P \times Leb$. ∎

Remark 6.3. In the above proof, the fact that the r. measure Λ is **ID** or even independently scattered is not important. In fact, similar methods can be used to prove a version of Theorem 6.2, where Λ is an arbitrary random measure and $\int f d\Lambda$ is defined as a limit, in some appropriate sense, of stochastic integrals of S_0-measurable simple functions.

VII. APPENDIX

PROOF OF PROPOSITION 2.4: First we note that (c) implies that $Q_0(\cdot, B)$ is σ-finite for every $B \epsilon \mathcal{B}$. To begin with we assume, in addition, that $X = R$ and $\mathcal{B} = \mathcal{B}(R)$.

Note, for every fixed $B \epsilon \mathcal{B}$, $Q_0(\cdot, B) \leq Q_0(\cdot, X) = \lambda_0(\cdot)$; therefore we can define

$$q_0(\cdot, B) = \frac{dQ_0(\cdot, B)}{d\lambda_0}. \tag{7.1}$$

Using uniqueness of the Radon-Nikodym derivative and the properties of Q_0, one can easily verify that q_0 satisfies the following properties:

(a′) If $\{B_j\} \subseteq \mathcal{B}$, $B_j \uparrow B$ then $q_0(\cdot, B_j) \uparrow q_0(\cdot, B)$ a.e. $[\lambda_0]$,

(b′) If $\{B_j\} \subseteq \mathcal{B}$, $B_j \downarrow B$, then $q_0(\cdot, B_j) \downarrow (\cdot, B)$ a.e. $[\lambda_0]$,

(c′) $q_0(\cdot, X) = 1$ a.e. $[\lambda_0]$.

Let D be any countable dense subset of $\mathbf{R}$; using (a'), (b') and (c') we can choose a set $M \epsilon \mathcal{A}$ such that $\lambda_0(M) = 0$ and if $t \notin M$, then $q_0(t, (-\infty, r_1]) \leq q_0(t, (-\infty, r_2])$, for $r_1, r_2 \epsilon D$, $r_1 \leq r_2$; and $\lim_{\substack{r \uparrow \infty \\ r \epsilon D}} q_0(t, (-\infty, r]) = 1$, $\lim_{\substack{r \downarrow -\infty \\ r \epsilon D}} q_0(t, (-\infty, r]) = 0$. Let, for $r_0 \epsilon D$,

$$N_{r_0} = \left\{ t \notin M : \lim_{\substack{r \downarrow r_0 \\ r \epsilon D}} q_0\,(t, (-\infty, r]) > q_0\,(t, (-\infty, r_0]) \right\},$$

then, by (b'), $\lambda(N_{r_0}) = 0$, for all $r_0 \epsilon D$. Set $N = M \cup \left(\bigcup_{r \epsilon D} N_r \right)$; and define, for every $(t, x) \epsilon T \times \mathbf{R}$,

$$F(t, x) = \left\{ \begin{array}{cc} G(x), & \text{if } t \epsilon N, \\ \lim_{\substack{r \downarrow x \\ r \epsilon D}} q_0\,(t, (-\infty, r]), & \text{if } t \notin N \end{array} \right\},$$

where G is an arbitrary prob. distribution function on $\mathbf{R}$. It is easy to see that, for every fixed t, $F(t,\cdot)$ is a prob. distribution function on $\mathbf{R}$ and, for every fixed x, $F(\cdot,x)$ is measurable on T. Thus, for each t, there exists a unique prob. measure $q(t,\cdot)$ on $\mathbf{R}$ satisfying

$$q(t,(-\infty,x]) = F(t,x),$$

for every $x\epsilon\mathbf{R}$. Now, using the standard monotone class theorem argument, we obtain that, for each $B\epsilon\mathcal{B}$, $q(\cdot,B)$ is measurable and that $q(\cdot,B) = q_0(\cdot,B)$ a.e. $[\lambda_0]$. Hence, we have that q satisfies (d) and (c) and, from (7.1),

$$Q_0(A,B) = \int_A q(t,B)\lambda_0(dt), \tag{7.2}$$

for all $A\epsilon\mathcal{A}$, $B\epsilon\mathcal{B}$.

Now let $(X,\mathcal{B})$ be standard Borel space; then one can find a Borel subset E of R such that $\mathcal{B}$ and $\mathcal{B}(E)$ are σ-isomorphic. We denote this isomorphism by τ, and we define

$$Q_0'(A,F) = Q_0(A,\tau^{-1}(F\cap E)),$$

for $A\epsilon\mathcal{A}$, $F\epsilon\mathcal{B}(\mathbf{R})$; then, from what we have proved above, we obtain $q' : T\times\mathcal{B}(\mathbf{R}) \longrightarrow [0,1]$ which satisfies (d) and (e); and, (7.2) with Q_0 replaced by Q_0' and q by q'. Finally, let

$$q(t,B) = q'(t,\tau(B)),$$

for every $t\epsilon T$, $B\epsilon\mathcal{B}$. Then clearly q satisfies (d) and (e); and, from (7.2), we also have

$$\begin{aligned} Q_0(A,B) &= Q_0'(A,\tau(B)) \\ &= \int_A q'(t,\tau(B))\,\lambda_0(dt) \\ &= \int_A q(t,B)\lambda_0(dt). \end{aligned}$$

Now, using Tulcea's theorem [**1**, p. 209] there exists a unique Q on $\mathcal{A}\times B$ such that (2.2) holds. The uniqueness of $q(\cdot,\cdot)$ as stated in the last part of the proposition now follows easily using the fact that $\mathcal{B}$ is countably generated. ∎

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